2017 Q1

(a) Note $L(\beta) = (Y - X\beta)^T (Y - X\beta) + \lambda_n \beta^T \beta$

$$\therefore \frac{\partial L}{\partial \beta} = -2X^T(Y - X\beta) + 2\lambda_n \beta$$

$$\therefore \frac{\partial^2 L}{\partial \beta \partial \beta^T} = 2X^TX + 2\lambda_n I_p \succ 0$$

(so $\lambda_n I_p \succeq 0$)

~~B~~ Since $I_p \succ 0$, and $X^TX \succeq 0$ a.s.

Hence L is to strictly convex and has a unique minimiser given by

$$\frac{\partial L}{\partial \beta} = 0 \quad \text{~~i.e.~~} \quad \therefore -2X^TY + 2X^TX\beta + 2\lambda_n I_p \beta = 0$$

$$\therefore \hat{\beta}_\lambda = (X^TX + \lambda_n I_p)^{-1} X^TY$$

(b) $\hat{\beta}_\lambda = \left(\frac{X^TX + \lambda I_p}{n}\right)^{-1} \frac{X^TY}{n}$

now $Y_i = \sum_{j=1}^{p} \beta_j X_{ij} + \varepsilon_i$ where $\varepsilon_i \overset{iid}{\sim} N(0,1)$
(independently of X)

$\therefore Y = X\beta + \varepsilon$ where $\varepsilon \sim N(\vec{0}, I_n)$ and $X \perp\!\!\!\perp \varepsilon$.

$$\therefore \frac{X^TY}{n} = \frac{X^TX}{n}\beta + \frac{X^T\varepsilon}{n}$$

Now note: $\frac{1}{n}[X^T\varepsilon]_i = \frac{1}{n}\sum_{j=1}^{n} X_{ji}\varepsilon_j \xrightarrow{P} 0$

By WLLN as $X_{ji}\varepsilon_j$ are iid products of independent $N(0,1)$

$(E X_{ji}\varepsilon_j = E X_{ji} E\varepsilon_j = 0$, $E(X_{ji}\varepsilon_j)^2 = E X_{ji}^2 E\varepsilon_j^2 = 1)$

and $\frac{1}{n}[X^TX]_{ij} = \frac{1}{n}\sum_{k=1}^{n} X_{ik} X_{jk} \xrightarrow{P} \begin{cases} 1 & \text{if } i=j \\ 0 & i\neq j \end{cases}$

by WLLN since $X_{ik}^2 \sim \chi^2_1$ with $E = 1$ $Var = 2$

whereas $X_{ik}X_{jk}$ has $E = 0$ $Var = 1$.

Hence, by ~~Stud~~ ~~CMT~~ continuous mapping thm,

$$\frac{1}{n}X^TY \xrightarrow{P} I_p\beta + 0 = \beta$$

$$\frac{1}{n}(X^TX + \lambda_n \cancel{\beta} I_p) \xrightarrow{P} I_p + I_p \lim \frac{\lambda_n}{n}$$

Hence $\hat{\beta}_\lambda \xrightarrow{P} \frac{\beta}{1 + \lim \frac{\lambda_n}{n}}$

$\therefore$ the answer is $\{\beta, \frac{\beta}{6}, 0\}$

2017 Q1

$$\text{(c)}\quad \sqrt{n}\left(\hat{\beta}_\lambda - \beta\right) = \sqrt{n}\left((X^TX + \lambda_n I_p)^{-1} X^T Y - \beta \right)$$

$$= \sqrt{n}\left[(X^TX + \lambda_n I_p)^{-1} (X^TX\beta + X^T\varepsilon) - \beta \right]$$

$$= \sqrt{n}\left[(X^TX + \lambda_n I_p)^{-1} (-\lambda_n I_p \beta + X^T\varepsilon) \right]$$

$$= \left(\frac{X^TX}{n} + \frac{\lambda_n}{n} I_p \right)^{-1} \left(\frac{X^T\varepsilon}{\sqrt{n}} - \frac{\lambda_n}{\sqrt{n}} I_p \beta \right)$$

Now, as $\frac{\lambda_n}{\sqrt{n}} \to 0$ or 5, $\quad \frac{\lambda_n}{n} = \frac{1}{\sqrt{n}}\left(\frac{\lambda_n}{n}\right) \to 0$

and so, & similarly to b, we have that

$$\frac{X^TX}{n} + \frac{\lambda_n}{n} I_p \xrightarrow{P} I_p$$

Also, as $[X^T\varepsilon]_i = \sum_{j=1}^{n} X_{ji}\varepsilon_j$ is an iid sum, by CLT

$$\frac{1}{\sqrt{n}}[X^T\varepsilon]_i \xrightarrow{d} N(0, 1) \qquad \left(E X_{ji}\varepsilon_j = 0,\ E\ {}^2 = 1 \right)$$

~~independent~~ uncorrelated → justify (quick matrix calculation)

and the components are ~~independent~~ uncorrelated so, by Multivariate CLT,

$$\frac{X^T\varepsilon}{\sqrt{n}} \xrightarrow{d} N(\vec{0}, I_p).$$

By Slutsky's theorem,

$$\sqrt{n}\left(\hat{\beta}_\lambda - \beta\right) \xrightarrow{d} N(\vec{0}, I_p) - \beta \lim \frac{\lambda_n}{\sqrt{n}}$$

So the answer is $\left\{ N(\vec{0}, I_p),\ N(-5\beta, I_p) \right\}$.

(d) MLE is unbiased but has higher variance
$\hat{\beta}_\lambda$ is biased but has lower variance

Asymptotically they are both unbiased ($\sqrt{n}$ consistent)
and have the same variance

In practice, ~~B~~ ~~for pr~~ $\hat{\beta}_\lambda$ might be better for prediction as $MSE = bias^2 + var^2$
and the reduction in variance may justify the ~~[illegible]~~ increase in bias.

For inference, $\hat{\beta}_{MLE}$ might be preferred as it is unbiased.

Also, if $p > n$, need ridge (otherwise singular $(\dots)^{-1}$)

2017 Q2

(a) Case 1: $a = 0$.

By class results, we know a Bayes estimator of θ cannot be unbiased unless it is equal to θ a.s.

$\therefore$ $\delta_a(X) = X$ cannot be a Bayes estimator

Case 2: $a \neq 0$.

Then $R(\theta, \delta_a) = E_\theta(X - a - \mu)^2$

$$= E_\theta(X-\mu)^2 - 2a E_\theta(X-\theta) + a^2$$
$$= 1 + a^2$$
$$> R(\theta, \delta_0) \quad \forall \theta.$$

$\therefore$ $\delta_a(X)$ is inadmissible so cannot be a Bayes estimator ($\delta_0 X$ has infinite Bayes risk).

(b) We prove that a can only be equal to 0 in this case.

Firstly we show that $\exists$ a sequence that works for $a = 0$.

Suppose ~~θ~~ $\pi(\theta) \sim N(\mu, \tau^2)$. Then

$\pi(\theta | X) \sim N\left(\frac{\frac{\mu}{\tau^2} + X}{\frac{1}{\tau^2} + 1}, \frac{1}{\frac{1}{\tau^2} + 1} \right)$ as per the examples from class.

$\therefore$ ~~$E\,\theta|X$~~ $\delta_n = E\,\theta | X = \frac{\frac{\mu}{\tau^2} + X}{\frac{1}{\tau^2} + 1}$

For our sequence, pick $\pi_n(\theta) \sim N(0, \tau_n^2 = \frac{1}{n^2})$

Then $\delta_{\pi_n}(X) = \dfrac{X}{\frac{1}{n^2}+1}$

$$\therefore E\left(\delta_{\pi_n}(X) - (X+0)\right)^2 = E\left[\frac{\frac{1}{n^2}X}{\frac{1}{n^2}+1}\right]^2$$

$$= \frac{1}{(n^2+1)^2} E X^2$$

$$= \frac{1}{(n^2+1)^2} E_{0,\tau^2=\frac{1}{n}}\left(1+\theta^2\right)$$

$$= \frac{1}{(n^2+1)^2} E\left(1+\tau_n^2\right)$$

$$= \frac{1+\frac{1}{n^2}}{(n^2+1)^2} \longrightarrow 0 \quad \text{as required.} \quad \square$$

Secondly, fix $a \neq 0$ and suppose for a contradiction that $\exists$ a sequence π_n

s.t. $E\left(\delta_{\pi_n}(X) - X - a\right)^2 \longrightarrow 0$ as $n \to \infty$

We already showed that X is an estimator with constant

risk $R(\theta, X) = 1$ so $r(\pi_n, X) = 1$ and therefore $r(\pi_n) \leq 1$

$\therefore \limsup_{n\to\infty} r(\pi_n) \leq 1$

also, $R(\theta, a+X) = 1+a^2$ so $r(\pi_n, a+X) = 1+a^2 > 1$.

But $r(\pi_n) = E\left(\delta_{\pi_n}(X) - \theta\right)^2$

$$= \underbrace{E\left(\delta_{\pi_n}(X) - X - a\right)^2}_{\longrightarrow 0 \text{ by assumption}} \cancel{+} - 2E\left(\delta_{\pi_n}(X) - X - a\right)\left(X + a - \theta\right) + \underbrace{E\left(X + a - \theta\right)^2}$$

2017 Q2

We will show that $E(\delta_{\pi_n}(X) - X - a)(X + a - \theta) \longrightarrow 0$

whence it will follow that $r(\pi_n) \longrightarrow 1 + a^2$, thus

giving the desired contradiction.

But by Cauchy Schwarz inequality

$$E(\delta_{\pi_n}(X) - X - a)(X + a - \theta) \leq \sqrt{E(\delta_{\pi_n}(X) - X - a)^2 \, E(X + a - \theta)^2}$$

$$= \sqrt{1 + a^2} \sqrt{E(\delta_{\pi_n}(X) - X - a)^2} \longrightarrow 0 \quad \text{by assumption } \square.$$

$$\sqrt{E[(\delta_{\pi_n}(X) - X - a)^2]} \to 0 \Rightarrow \sqrt{E(\ldots)^2} \to 0$$

$\therefore \; \|\delta_n(X) - X - a\|_2 \to 0$

But $\|\delta_n(X) - X - a\|$

$= \|\delta_n(X) - \theta - (X + a - \theta)\|_2 \geq \big| \|\delta_n - \theta\|_2 - \|X + a - \theta\|_2 \big|$

reverse triangle inequality.

In general ~~$X_n \Rightarrow X$~~ $x_n \xrightarrow{L_p} X$

$\Longrightarrow E|X_n|^p \to E|X|^p$

$\|X_n - X\|_p \geq \big| \|X_n\|_p - \|X\|_p \big|$

2017 Q3

(a) Let s_N be the scaling.

Note

$$s_N(\bar{\theta}_N - \theta) = s_N\left(\frac{1}{m}\sum_{j=1}^{m}\hat{\theta}_n(j) - \theta\right)$$

$$= \frac{1}{m}\sum_{j=1}^{m} s_N(\hat{\theta}_n(j) - \theta)$$

So $s_N = r_n = r_{\frac{N}{m}}$, gives the appropriate scaling since

$$r_n(\bar{\theta}_N - \theta) = \frac{1}{m}\sum_{j=1}^{m}\underbrace{r_n(\hat{\theta}_n(j) - \theta)}_{\xrightarrow{d} G}$$

$$\xrightarrow{d} \frac{1}{m}\sum_{j=1}^{m} G_j \quad \text{where } G_1, \ldots, G_m \overset{iid}{\sim} G$$

the limiting distn is $\bar{G}$, the distn of the average of m iid copies of G.

(b) $R(\theta, \hat{\theta}) = E(\hat{\theta} - \theta)^2$

$\therefore \; N^{2\gamma}R(\theta,\hat{\theta}) = E\left[N^{\gamma}(\hat{\theta}-\theta)\right]^2 \longrightarrow E\,G^2$ by UI

If $r_N = N^{\gamma}$, $s_N = r_n = r_{\frac{N}{m}} = \left(\frac{N}{m}\right)^{\gamma}$

$$\therefore \quad \left(\frac{N}{m}\right)^{2\gamma} R(\theta,\bar{\theta}) = E\left[\left(\frac{N}{m}\right)^{\gamma}(\bar{\theta}-\theta)\right]^2 \longrightarrow E\left[\frac{1}{m}\sum_{j=1}^{m} G_j\right]^2 \quad \text{by HI}$$

$$= \frac{1}{m} E G^2$$

$$\therefore \quad N^{2\gamma} R(\theta,\bar{\theta}) \longrightarrow m^{2\gamma-1} E G^2 .$$

Thus the relative efficiency is asymptotically

$$\frac{R(\theta,\hat{\theta})}{R(\theta,\bar{\theta})} = \frac{N^{2\gamma} R(\theta,\hat{\theta})}{N^{2\gamma} R(\theta,\bar{\theta})} \longrightarrow m^{1-2\gamma}$$

(c) Thus, we see for

$\gamma > \frac{1}{2}$ $\quad \hat{\theta}$ has better risk.

$\gamma < \frac{1}{2}$ $\quad \bar{\theta}$ has better risk.

2017 Q4

(a) $X_1, \dots, X_n \overset{iid}{\sim} F \quad \therefore \quad F(X_1), \dots, F(X_n) \overset{d}{=} U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

$$\therefore \quad D_n = \sup_{x \in \mathbb{R}} \left| \frac{1}{n} \sum \mathbb{1}_{\{X_i \le x\}} - F(x) \right|$$

$$= \sup_{x \in \mathbb{R}} \left| \frac{1}{n} \sum \mathbb{1}_{\{F(X_i) \le F(x)\}} - F(x) \right|$$

$$\overset{d}{=} \sup_{u \in [0,1]} \frac{1}{n} \left| \sum \mathbb{1}_{\{U_i \le u\}} - n u \right| \qquad \square$$

(b) Under H_0,

$$f_\theta(X) = \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}_{\{X_i < \theta\}} = \frac{1}{\theta^n} \mathbb{1}_{\{X_{(n)} < \theta\}}$$

$\therefore$ $X_{(n)}$ is sufficient by Neyman Fisher factorisation criterion

But [illegible] $X_{(n)}$ is a function of ~~$X_{(n)}$~~ (J, X_J)

(as $X_{(n)} = X_J$). Therefore (J, X_J) is sufficient $\square$

$$\left(\text{to be precise } f_\theta(x) = \overbrace{\frac{1}{\theta^n} \mathbb{1}\{X_J < \theta\}}^{\text{function of } \theta \text{ and } (J, X_J)} \right)$$

Next, we compute $P(X_i < x_i \ \forall i \ne j \mid J = j, X_j = t) = P(X_i < x_i \ \forall i \ne j \mid X_j = t, X_i < t \ \forall i \ne j)$

$$= \frac{P(X_i < x_i \ \forall i \ne j, X_i < t \ \forall i \ne j)}{P(X_i < t \ \forall i \ne j)} = \prod \left(\frac{x_i}{\theta}\right) \Big/ \left(\frac{t}{\theta}\right)^{n-1} = \prod \left(\frac{x_i}{t}\right) \qquad \square$$

$P(X_i \in (x_i, x_i + dx_i) \ \forall i \ne j \mid J = j,$ [illegible] $X_j \in (t, dt))$

$$= \frac{P(X_i \in (x_i, x_i + dx_i) \ \forall i \ne j, X_j \in (t, t + dt), J = j)}{P(X_j \in (t, t + dt), J = j)}$$

$$= \frac{\left(\prod_{i\neq j} \frac{dx_i}{\theta}\right) \cancel{dt} \cdot \frac{dt}{\theta} \cdot \mathbb{1}\{t > x_i \forall i\}}{P(X_j \in (t, t+dt),\ X_i < t\ \forall i)}$$

$$= \frac{\left(\prod_{i\neq j} \frac{dx_i}{\theta}\right)\left(\frac{dt}{\theta}\right) \mathbb{1}\{t > x_i \forall i\}}{\left(\frac{dt}{\theta}\right) \cdot \left(\frac{t}{\theta}\right)^{n-1}}$$

$$= \prod_{i\neq j} \left(\frac{dx_i}{t}\right) \mathbb{1}\{x_i < t\}$$

$$\therefore f_{X_i : i\neq j \mid X_j = t, J=j}(x_i) = \prod_{i \neq j} \frac{1}{t} \mathbb{1}\{x_i < t\}$$

$\therefore$ $X_i : i \neq j$ are iid $U(0,t)$ given $(J, X_J) = (j, t)$. $\square$

(c) Under H_0, $D_n = \frac{1}{n} \sup_{x \in [0,\theta]} \left| \sum^{n} \mathbb{1}\{X_i \leq x\} - \frac{x}{\theta_0} \right|$

will have a certain distribution, which can be computed, for example, numerically by simulation. If the observed data yields a value of D_n that is large compared to the distn of D_n under the null, reject.

~~Or choose $X_{(n)}$ and remove the observation~~

If θ_0 is unknown, choose $X_{(n)}$ and remove this observation from D_n.

2017 Q5

Under H_0, $p(\vec{X}) = \frac{1}{8}$ for $\vec{X} \in \{0,1\}^3$

(a) Under the alternative θ,

$$p_\theta(x_1,x_2,x_3) = \sum_{i=1}^{3} P(x_1,x_2,x_3 \mid I = i)\ \frac{1}{3}$$

$$= \frac{1}{3}\sum_{i=1}^{3} \frac{1}{4}\cdot\left(\frac{1}{2}+\theta\right)^{x_i}\left(\frac{1}{2}-\theta\right)^{1-x_i}$$

$$= \frac{1}{12}\sum_{i=1}^{3}\left(\frac{1}{2}+\theta\right)^{x_i}\left(\frac{1}{2}-\theta\right)^{1-x_i}$$

Note that if $\theta = 0$, this reduces to H_0.

Consider the alternative $H_1: \theta = \theta_1 > 0$.

By NP lemma, an MP test exists of the form:

$$\phi(\vec{X}) = \begin{cases} 1 & \text{if } p_{\theta_1}(\vec{X}) > k\, p_0(\vec{X}) \\ 0 & \text{if } p_{\theta_1}(\vec{X}) < k\, p_0(\vec{X}) \end{cases}$$

$E_{\theta=0}\,\phi(\vec{X}) = \alpha$.

Note
$$\frac{p_{\theta_1}(\vec{X})}{p_{\theta_0}(\vec{X})} = \frac{1}{3}\sum_{i=1}^{3}(1+2\theta_1)^{x_i}(1-2\theta_1)^{1-x_i}$$

$$= \frac{1}{3}\sum_{i=1}^{3}(1+2\theta_1)\,\mathbb{1}\{x_i=1\} + (1-2\theta_1)\,\mathbb{1}\{x_i=0\}$$

$$= \frac{1}{3}\sum_{i=1}^{3}\left\{1 + 2\theta_1(2x_i - 1)\right\}$$

$$= 1 + \frac{2\theta_1}{3}\sum_{i=1}^{3}(2x_i - 1)$$

$$= 1 + 4\theta_1 \bar{x} - 2\theta_1$$

$$= 1 + 2\theta_1(2\bar{x} - 1) \quad . \quad = 1 + 2\theta_1\left(\frac{2}{3}\sum_1^3 x_i - 1\right)$$

Now, want to fix k s.t.

$$E_{\theta=0}\,\phi(x) = \alpha$$

$$\therefore\ P_{\theta=0}(1 + 2\theta_1(2\bar{x}-1) > k) + \nu P_{\theta=0}(1 + 2\theta_1(2\bar{x}-1) = k) = \alpha \,.$$

$$\therefore\ P_{\theta=0}\left(\cancel{2\theta_1}\, 2\sum_1^3 x_i > \frac{k-1}{2\theta_1} + 1\right) + \nu P_{\theta=0}\left(2\sum_1^3 x_i = \frac{k-1}{2\theta_1} + 1\right) = \alpha$$

$$\therefore\ P\left(\text{Bin}(3,\tfrac{1}{2}) > \frac{k-1}{4\theta_1} + \frac{1}{2}\right) + \nu P\left(\text{Bin}(3,\tfrac{1}{2}) = \frac{k-1}{4\theta_1} + \frac{1}{2}\right) = \alpha$$

$\therefore$ must pick k such that $\frac{k-1}{4\theta_1} + \frac{1}{2}$ equals the unique integer z such that $P(\text{Bin}(3,\tfrac{1}{2}) > z) \leq \alpha \overset{\text{strict}}{\leq} P(\text{Bin}(3,\tfrac{1}{2}) \geq z)$

and pick ~~ν s.t.~~, $\nu = \dfrac{\alpha - P(\text{Bin}(3,\frac{1}{2}) > z)}{P(\text{Bin}(3,\frac{1}{2}) = z)}$

$$\therefore\ \phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_1^3 x_i > z \\ \nu & \text{if } \sum_1^3 x_i = z \\ 0 & \text{o/w} \end{cases}$$

As ϕ is free of the alternative, it follows that it is UMP for $\theta = 0$ vs $\theta > 0$. □

2017 Q5

(b) Now, under the alternative,

$$p_\theta(x_1,x_2,x_3) = \sum_{i=1}^{3} \sum_{j\neq 0}^{2} P(x_1,x_2,x_3 \mid I=i, J=j) \frac{1}{6}$$

$$= \frac{1}{6} \sum_{i=1}^{3} \sum_{j\neq 0}^{2} \frac{1}{4} \left(\tfrac{1}{2}+j\theta\right)^{x_i} \left(\tfrac{1}{2}-j\theta\right)^{1-x_i}$$

$$= \frac{1}{6} \cdot \frac{1}{8} \sum_{j\neq 0}^{2} \sum_{i=1}^{3} (1+2j\theta)^{x_i} (1-2j\theta)^{1-x_i}$$

$$= \frac{1}{6} \cdot \frac{1}{8} \sum_{j\neq 0}^{2} \sum_{i=1}^{3} 1 + 2j\theta(2x_i - 1)$$

$$= \frac{1}{6} \cdot \frac{1}{8} \sum_{j\neq 0}^{2} \left(3 + 4j\theta \sum^{3} x_i - 6j\theta \right)$$

$$= \frac{1}{8}$$

$$\therefore \quad \frac{p_\theta(\vec{x})}{p_{\theta_0}(\vec{x})} = 1 \qquad \forall \theta > 0$$

$\therefore$ UMP test is $\phi \equiv \alpha$.

2017 Q6

→ More cleanly, let $R_1, \dots, R_n$ be the ranks of $X_1, \dots, X_n$ $(R_i = j$ if $X_{(j)} = i)$
then ~~&~~ $X_i > X_j \Leftrightarrow R_i > R_j$

(a) To begin with, we reshuffle the sum in A_n :

$$\sum_{i \neq j} \mathbb{1}\{(\pi(i)-\pi(j))(\sigma(i)-\sigma(j)) > 0\} =$$

$$= \sum_{\substack{i' \neq j' \\ i=\pi^{-1}(i'),\, j=\pi^{-1}(j')}} \mathbb{1}\{(i'-j')(\sigma(\pi^{-1}(i')) - \sigma(\pi^{-1}(j'))) > 0\}$$

$$= \sum_{i \neq j} \mathbb{1}\{(i^*-j)(\tau(i) - \tau(j)) > 0\} \qquad ①$$

where $\tau = \sigma \circ \pi^{-1}$.

Clearly, as π is uniform on S_n, so is π^{-1} and $\sigma \circ \pi^{-1}$.

Secondly, define a new permutation ρ ~~[illegible]~~ on S_n by

$$\rho : i \longmapsto j \quad \text{iff} \quad X_i = X_{(j)}$$

As $X_1, \dots, X_n \overset{iid}{\sim} U(0,1)$, clearly ρ is uniform on S_n.

(as each ~~(arrangement)~~ permutation of our sample into ordered statistics is equally likely).

Now, using a similar reshuffling argument to above, we have

$$\sum_{i \neq j} \mathbb{1}\{(X_i - X_j)(Y_i - Y_j) > 0\} = \sum_{\substack{i' \neq j' \\ \rho(i')=i,\, \rho(j')=j}} \mathbb{1}\{(X_{\rho(i')} - X_{\rho(j')})(Y_{\rho(i')} - Y_{\rho(j')}) > 0\}$$

$$= \sum_{i \neq j} \mathbb{1}\{(X_{(i)} - X_{(j)})(Y_{\rho(i)} - Y_{\rho(j)}) > 0\}$$

$$= \sum_{i \neq j} \mathbb{1}\{(i-j)(Y_{\rho(i)} - Y_{\rho(j)}) > 0\} \quad \text{a.s.} \quad (\text{as } P(Y_{(i)} = X_{(j)}) = 0)$$

And as the X_i's are independent of the Y_i, also ρ is independent of the Y_i and the distribution of ~~$Y_{\rho(i)} - Y_{\rho(j)}$~~ $Y_{\rho(1)}, \dots, Y_{\rho(n)}$ is the same as $\tilde{Y}_1, \dots, \tilde{Y}_n \overset{iid}{\sim} U(0,1)$. Hence

$$B_n \overset{d}{=} \frac{1}{n(n-1)} \sum_{i \neq j} \mathbb{1}\{(i-j) \cdot \operatorname{sign}(\tilde{Y}_i - \tilde{Y}_j) > 0\}$$

Defining the permutation π on S_n by

$\pi(i) = j$ iff $\tilde{Y}_i = \tilde{Y}_{(j)}$, we find

$$B_n \overset{d}{=} \frac{1}{n(n-1)} \sum_{i \neq j} \mathbb{1}\{(i-j) \cdot \operatorname{sign}(\tilde{Y}_{(\pi(i))} - \tilde{Y}_{(\pi(j))}) > 0\}$$

$$= \frac{1}{n(n-1)} \sum_{i \neq j} \mathbb{1}\{(i-j)(\pi(i) - \pi(j)) > 0\}$$

where π is independent of X and ρ.

This clearly has the same distn. as I, whence the result follows. □

2017 Q6

(b) $2B_n$ is a U-Statistic with Kernel

$$h\left(\begin{pmatrix}X_1\\Y_1\end{pmatrix},\begin{pmatrix}X_2\\Y_2\end{pmatrix}\right) = \mathbb{1}\{(X_1-X_2)(Y_1-Y_2)>0\}$$

$$\therefore\ E\,h\left(\begin{pmatrix}X_1\\Y_1\end{pmatrix},\begin{pmatrix}X_2\\Y_2\end{pmatrix}\right) = P\left((X_1-X_2)(Y_1-Y_2)>0\right)$$

$$= P(X_1>X_2, Y_1>Y_2) + P(X_1<X_2, Y_1<Y_2)$$

$$= \frac{1}{4} + \frac{1}{4} \qquad (\text{independence})$$

$$= \frac{1}{2}$$

Now compute

$$\zeta_1 = \mathrm{Cov}\left[h\left(\begin{pmatrix}X_1\\Y_1\end{pmatrix},\begin{pmatrix}X_2\\Y_2\end{pmatrix}\right), h\left(\begin{pmatrix}X_1\\Y_1\end{pmatrix},\begin{pmatrix}X_3\\Y_3\end{pmatrix}\right)\right]$$

$$= E\left[\mathbb{1}\{(X_1-X_2)(Y_1-Y_2)\}\mathbb{1}\{(X_1-X_3)(Y_1-Y_3)>0\}\right] - \left(\frac{1}{2}\right)^2$$

$$= P(X_1>X_2, Y_1>Y_2, X_1>X_3, Y_1>Y_3) + P(X_1>X_2, Y_1>Y_2, X_1<X_3, Y_1<Y_3)$$
$$+ P(X_1<X_2, Y_1<Y_2, X_1>X_3, Y_1>Y_3) + P(X_1<X_2, Y_1<Y_2, X_1<X_3, Y_1<Y_3) - \frac{1}{4}$$

$$= 2P(X_1>X_2, X_1>X_3, Y_1>Y_2, Y_1>Y_3) + 2P(X_3>X_1>X_2, Y_3>Y_1>Y_2) - \frac{1}{4}$$

$$= 2P(\max(X_1,X_2,X_3)=X_1)^2 + 2P(X_3>X_1>X_2)^2 - \frac{1}{4}$$

$$= 2\cdot\frac{1}{3^2} + 2\cdot\frac{1}{6^2} - \frac{1}{4} = \frac{8+2-9}{36} = \frac{1}{36}$$

~~$= \frac{5}{8} - \frac{1}{4}$~~

By class results,

$$\sqrt{n}\left(2B_n - \frac{1}{2}\right) \xrightarrow{d} \cancel{N(0,1)} . \quad \square$$

$$N(0, 2^2 \frac{1}{36}) = N(0, \frac{1}{9}).$$

2016 Q1

(a) Consider the prior $\pi(\lambda) \propto \lambda^{\alpha-1} e^{-\beta\lambda}$.

Then $\pi(\lambda|x) \propto L(\lambda; x)\, \pi(\lambda)$

$$\propto e^{-n\lambda} \lambda^{\Sigma x_i} \lambda^{\alpha-1} e^{-\beta\lambda}$$

$$= \lambda^{\Sigma x_i + \alpha - 1} e^{-(\beta+n)\lambda}$$

$\therefore$ $\lambda | x \sim \text{Gamma}(\alpha + \Sigma x_i, \beta + n)$

$\therefore$ Bayes estimator is $\delta(x) = \frac{\alpha + \Sigma X_i}{\beta + n}$ (posterior mean)

~~and Bayes risk is $r(\pi, \delta) =$~~ [illegible] ~~$\frac{\alpha + \Sigma X_i}{(\beta+n)^2}$~~

The risk of this estimator is:

$$R(\lambda, \delta) = E(\delta(x) - \lambda)^2$$

$$= \text{Bias}\, \delta(x)^2 + \text{Var}\, \delta(x)$$

$$= \left(\frac{\alpha + n\lambda}{\beta + n} - \lambda\right)^2 + \frac{1}{(\beta+n)^2} n\lambda$$

$$= \left(\frac{\alpha - \beta\lambda}{\beta + n}\right)^2 + \frac{n\lambda}{(\beta+n)^2}$$

$$= \frac{\alpha^2 - 2\alpha\beta\lambda + n\lambda + \beta^2\lambda^2}{(n+\beta)^2}$$

And the Bayes risk is therefore :

$$r(\pi,\delta) = E\,R(\theta,\delta) =$$

$$= \frac{\alpha^2 - 2\alpha\beta\left(\frac{\alpha}{\beta}\right) + n\left(\frac{\alpha}{\beta}\right) + \beta^2\left(\frac{\alpha}{\beta^2} + \frac{\alpha^2}{\beta^2}\right)}{(n+\beta)^2}$$

$$= \frac{\cancel{\alpha^2 - 2\alpha^2 + (\alpha^2+n)\frac{\alpha}{\beta} + \frac{\alpha^2}{\beta}} + n\frac{\alpha}{\beta} + \alpha + \alpha^2}{(n+\beta)^2} = \frac{n\frac{\alpha}{\beta} + \alpha}{(n+\beta)^2} = \frac{\alpha}{\beta(n+\beta)}$$

$$= \frac{-\cancel{\alpha^2} + (n+1)\frac{\alpha}{\cancel{\beta}} + \frac{\cancel{\alpha^2}}{\beta}}{(n+\beta)^2} = \frac{(n+\beta)\alpha}{(n+\beta)^3} \qquad \left(= \frac{\alpha}{\beta(\beta+n)} \right)$$

$$= \frac{-\beta\alpha^2 + (n+1)\alpha + \alpha^2}{\beta(n+\beta)^2} \qquad ①$$

Therefore we consider the sequence of priors ~~π_n~~ $\pi_n(\lambda) = \mathrm{Gamma}(\alpha_n, \beta_n)$

where ~~$\alpha_n = n\ \forall n$~~ $\nearrow \infty$ as $n \to \infty$ ~~$\beta_n = \frac{1}{n}$ 1 for~~ ~~as $n \to \infty$~~

$\alpha_n = 1\ \forall n$, $\beta_n = \frac{1}{n} \to 0$ as $n \to \infty$

From I, $r(\pi_n, \delta) \longrightarrow \infty$ as $n \to \infty$.

~~But the estimator~~ $\therefore$ the minimax risk is ∞.

(in particular, $\delta_0(X) = \frac{\sum X_i}{n}$ has risk $R(\theta,\delta_0) = \frac{1}{n^2} E(\sum X_i - n\lambda)^2 = \frac{1}{n^2}\mathrm{Var}\sum X_i$
$= \frac{\lambda}{n}$

and so $\lim \sup\limits_{\lambda\in(0,\infty)} R(\theta,\delta_0) = \infty = \lim\limits_{n\to\infty}(r(\pi_n)\ \delta)$ so δ_0 is minimax).

2016 Q1

(b) By a result from homeworks, the Bayes estimator under weighted squared error loss ~~$\frac{(\delta(X)-\theta)^2}{w(\theta)}$ is E~~

$$w(\theta)(\delta(X)-\theta)^2 \quad \text{is} \quad \frac{E(w(\theta)\,\theta \mid X)}{E(w(\theta) \mid X)}.$$

Therefore, in this case, the Bayes estimator is

$$\delta_{\pi}(X) = \frac{1}{E[\frac{1}{\lambda} \mid X]}$$

Using the Gamma (α, β) prior from (i),

$$E\left[\frac{1}{\lambda} \Big| X\right] = \int_0^\infty \frac{1}{\lambda} \cdot \frac{(n+\beta)^{\alpha+\sum X_i}}{\Gamma(\alpha+\sum X_i)} \lambda^{\alpha+\sum X_i - 1} e^{-(\beta+n)\lambda} \, d\lambda$$

$$= \frac{(n+\beta)^{\alpha+\sum X_i}}{\Gamma(\alpha+\sum X_i)} \cdot \frac{\Gamma(\alpha+\sum X_i - 1)}{(n+\beta)^{\alpha+\sum X_i - 1}} \quad \text{(recognise a Gamma density)}$$

$$= \frac{n+\beta}{\alpha - 1 + \sum X_i}$$

$$\therefore \quad \delta_{\pi}(X) = \frac{\alpha - 1 + \sum X_i}{n+\beta}$$

(same estimator as in (i) but replacing α by $\alpha - 1$).

By the same calculation as in (i), this has risk

$$R(\lambda, \delta) = \frac{(\alpha-1)^2 - 2(\alpha-1)\beta\lambda + n\lambda + \beta^2\lambda^2}{\lambda(n+\beta)^2}$$

~~& and Bayes risk~~

~~$r(\pi, \delta) = \frac{\beta(\alpha-1)^2 + (n-?)(\alpha-1) + (\alpha-1)^2}{\beta(n+\beta)^{2n}}$~~

$$= \frac{(\alpha-1)^2 \frac{1}{\lambda} - 2(\alpha-1)\beta + n + \beta^2\lambda}{(n+\beta)^2}$$

and noting that $E \frac{1}{\text{Gamma}(\alpha;\beta)} = \int_0^\infty \frac{1}{x} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx = \frac{\beta}{\alpha-1}$ if $\alpha > 1$,

the Bayes risk is

$$r(\pi,\delta) = \frac{\beta(\alpha-1) - 2(\alpha-1)\beta + n + \beta^2 \frac{\alpha}{\beta}}{(n+\beta)^2}$$

$$= \frac{-(\alpha-1)\beta + n + \alpha\beta}{(n+\beta)^2}$$

$$\cancel{= \frac{\alpha(1-\beta)+\beta+n}{(n+\beta)^2}}$$

$$= \frac{n+\beta}{(n+\beta)^2} \qquad r_\pi = \frac{1}{\beta+n} \quad \text{if } \alpha \geq 1$$

Now we choose the sequence $\alpha_k = 2 \ \forall k$, $\beta_k = \frac{1}{k} \to 0$ as $k \to \infty$

$\therefore r(\pi_k, \delta) \longrightarrow \frac{1}{n}$

But with $\delta_0 = \frac{\sum X_i}{n}$,

$$R(\lambda, \delta_0) = E\frac{(\bar{X}-\lambda)^2}{\lambda} = \frac{1}{\lambda} \cdot \frac{1}{n^2} E(\sum X_i - n\lambda)^2 = \frac{1}{\lambda n^2} \text{Var}(\sum X_i) = \frac{1}{n}$$

$$\therefore \sup_\lambda R(\lambda, \delta_0) = \frac{1}{n} = \lim_{k\to\infty} r(\pi_k)$$

By class results, δ_0 is minimax and the

minimax risk is therefore $\frac{1}{n}$.

2016 Q2

(a) $$\frac{dP_n}{d\mu_n} = \prod_{i=1}^{n} e^{-X_i} = \exp\{-\Sigma X_i\}$$

$$\frac{dQ_n}{d\mu_n} = \prod_{i=1}^{n} \theta_i e^{-\theta_i X_i} = (\Pi \theta_i) \exp\{-\Sigma \theta_i X_i\}$$

$$\therefore \frac{dQ_n}{dP_n} = (\Pi \theta_i) \exp\{-\Sigma(\theta_i - 1) X_i\}$$

$$\therefore \log \frac{dQ_n}{dP_n} = \Sigma \log \theta_i - \Sigma(\theta_i - 1) X_i$$

$$= \sum_{i=1}^{n} -(\theta_i - 1) X_i + \overbrace{\underbrace{\log(1 + (\theta_i - 1))}_{(\theta_i-1) \mp \frac{(\theta_i-1)^2}{2} \geq \leq (\theta_i-1) - \frac{(\theta_i-1)^2}{2} + \frac{(\theta_i-1)^3}{3} \cdot O((\theta_i-1)^3)}}^{= (\theta_i-1) \mp \frac{(\theta_i-1)^2}{2} + \frac{(\theta_i-1)^3}{3}}$$

$$= \sum_{i=1}^{n}\{-(\theta_i - 1) X_i + (\theta_i - 1)\} + \sum_{i=1}^{n} - \frac{(\theta_i-1)^2}{2} + \sum_{i=1}^{n} o((\theta_i - 1)^2)$$

Now let $K = \sum_{1}^{\infty} (\theta_i - 1)^2 < \infty$.

Then $-\Sigma \frac{(\theta_i-1)^2}{2} \longrightarrow -\frac{K}{2}$ and $\sum_{i=1}^{n} o((\theta_i-1)^2) \longrightarrow \tilde{K} < \infty$.

On the other hand, $E_{P_n}[-(\theta_i - 1) X_i + (\theta_i - 1)] = 0$

$\mathrm{Var}_{P_n}(-(\theta_i - 1) X_i + (\theta_i - 1)) = (\theta_i - 1)^2$.

$\therefore \mathrm{Var}_{P_n}\left(\sum_{1}^{n} -(\theta_i-1)X + (\theta_i - 1)\right) = \sum_{1}^{n} (\theta_i - 1)^2 \leq K \quad \forall n$

$\therefore P_n\left(\left|\sum_{1}^{n} -(\theta_i-1)X_i + (\theta_i-1)\right| > M\right) \leq \frac{K}{M^2} \quad \forall n$

$\therefore \sup_{n} P_n(|\Sigma -(\theta_i-1)X_i - (\theta_i-1)| > M) \longrightarrow 0$ as $M \to \infty$ (tightness)

$\therefore$ along a subsequence,

$$\log \frac{dQ_n}{dP_n} \xrightarrow{d} Z - \frac{k}{2} + \tilde{k} \quad \text{, by Prohorov's thm.}$$

for some random variable Z.

$$\therefore \quad \frac{dQ_n}{dP_n} \xrightarrow{d} e^{Z - \frac{k}{2} + \tilde{k}} \qquad \text{(CMT) along a subsequence}$$

and as $P(e^{Z - \frac{k}{2} + \tilde{k}} = 0) = 0$, $P_n \triangleleft Q_n$ by Le Cam's 1st lemma $\square$

(3) $$\frac{dP_n}{dQ_n} = \frac{\binom{n}{X} p_n^X (1-p_n)^{n-X}}{\left(\frac{\lambda^X e^{-\lambda}}{X!}\right)}$$

$$= \binom{n}{X} X! \, p_n^X (1-p_n)^{n-X} \lambda^{-X} e^{+\lambda}$$

$$= \frac{n!}{(n-X)!} \left(\frac{p_n}{1-p_n}\right)^X (1-p_n)^n \lambda^{-X} e^{\lambda}$$

$$\sim \sqrt{\frac{n}{n-X}} \frac{n^n e^{-n}}{(n-X)^{n-X} e^{-n+X}} \cdot \left(\frac{p_n}{1-p_n}\right)^X \left(1 - \frac{np_n}{n}\right)^n \lambda^{-X} e^{\lambda} \quad \text{(Stirling)}$$

$$\sim \left(\frac{n}{n-X}\right)^n e^{-X} (n-X)^X \left(\frac{p_n}{1-p_n}\right)^X \left(1 - \frac{np_n}{n}\right)^n \lambda^{-X} e^{\lambda}$$

$$\sim \left(1 + \frac{X}{n-X}\right)^n e^{-X} \left(\frac{np_n}{1-p_n} - X\frac{p_n}{1-p_n}\right)^X \left(1 - \frac{np_n}{n}\right)^n \lambda^{-X} e^{\lambda}$$

$$\sim 1 \quad \text{as } n \to \infty$$

since $np_n \to \lambda$ $\therefore$ $\left(1 - \frac{np_n}{n}\right)^n \to e^{-\lambda}$,

$$\left(1 + \frac{X}{n-X}\right)^n = \left(1 + \frac{1}{n}\left(\frac{Xn}{n-X}\right)\right)^n \to e^X$$

2016 Q2

and $\frac{np_n}{1-p_n} - x\frac{p_n}{1-p_n} \longrightarrow \lambda - X \cdot 0 = \lambda$

Hence $\frac{dP_n}{dQ_n} \xrightarrow{d} 1$

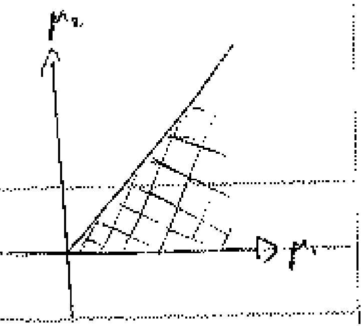
and so $P_n \Delta Q_n$.

(c) Let $A_n = \left\{ \frac{k}{\sqrt{n}} : k \in \mathbb{Z} \right\}$.

Then A_n is countable $\therefore$ $Q_n(A_n) = 0 \quad \forall n$

however $P_n(A_n) = 1 \quad \forall n \quad \therefore \quad P_n \not\Delta Q_n$

2016 Q4



(a) $L(\mu_1,\mu_2;X,Y) = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}(X-\mu_1)^2 - \frac{1}{2}(Y-\mu_2)^2\right\}$

$\therefore \ell(\mu_1,\mu_2;X,Y) =$ ~~[illegible]~~ $-\frac{1}{2}(X-\mu_1)^2 - \frac{1}{2}(Y-\mu_2)^2 +$ constant.

This is clearly maximal at ~~$(\frac{\partial}{\partial\mu_1},\frac{\partial}{\partial\mu_2}) = (X,Y)$~~ $(\mu_1,\mu_2) = (X,Y)$

Case 1:

$\therefore$ if $X \geq Y \geq 0$, then $(\hat{\mu}_1,\hat{\mu}_2) = (X,Y)$, is the MLE.

Case 2:

If, on the other hand, $X < 0$ and $Y < 0$, then

ℓ is decreasing in $\mu_1 \in [0,\infty)$ and in $\mu_2 \in [0,\infty)$

$\therefore$ ℓ is maximal at the endpoint $(\mu_1,\mu_2) = (0,0)$ and so

in this case $(\hat{\mu}_1,\hat{\mu}_2) = (0,0)$.

Case 3:

Now suppose $X \geq 0$, $Y < 0$. Then the term $-\frac{1}{2}(Y-\mu_2)^2$ is

decreasing in $\mu_2 \in [0,\infty)$ while the term $-\frac{1}{2}(X-\mu_1)^2$ is

minimal at $X = \mu_1$ $\therefore$ $(\hat{\mu}_1,\hat{\mu}_2) = (X,0)$.

Case 4: $X < Y$, $Y \geq 0$.

Let $\theta_1 = \mu_1 + \mu_2$, $\theta_2 = \mu_1 - \mu_2$ (so that $\theta_2 \geq 0$ and $\theta_1 \geq \theta_2$)

so $\mu_1 = \frac{\theta_1+\theta_2}{2}$, $\mu_2 = \frac{\theta_1-\theta_2}{2}$

Then $-\frac{1}{2}(X-\mu_1)^2 - \frac{1}{2}(Y-\mu_2)^2 = -\frac{1}{2}(X-\frac{\theta_1+\theta_2}{2})^2 - \frac{1}{2}(Y-\frac{\theta-\theta_2}{2})^2$

$$= -\frac{1}{4}(X+Y-\theta_1)^2 - \frac{1}{4}(X-Y-\theta_2)^2 =: \tilde{l}(\theta_1,\theta_2)$$

Now split into sub-cases

Case 4.1: $X<Y,\ Y\geq 0,\ X>-Y$

Then $X-Y<0$ and $X+Y>0$.

$\therefore\ \tilde{l}(\theta_1,\theta_2)$ is maximal at $(\theta_1,\theta_2)=(X+Y,0)$,

since $-(X+Y-\theta_1)^2$ is maximal at $\theta_1=X+Y$, whereas

$-(X-Y-\theta_2)^2$ is decreasing on $\theta_2\geq 0$, so it is maximal at

the endpoint, $\theta_2=0$.

By invariance of MLE, $(\hat{\mu}_1,\hat{\mu}_2)=\left(\frac{X+Y}{2},\frac{X+Y}{2}\right)$.

Case 4.2: $X<Y,\ Y\geq 0,\ X<-Y$.

Then $X-Y<0$ and $X+Y<0$.

$\therefore\ \tilde{l}(\theta_1,\theta_2)$ is maximal at $(\theta_1,\theta_2)=(0,0)$,

as both quadratic terms are decreasing in $\theta_1\in[0,\infty)$ and $\theta_2\in[0,\infty)$.

2016 Q4

Putting the pieces together,

$$(\hat{\mu}_1, \hat{\mu}_2) = \begin{cases} (X, Y) & \text{if } X \geq Y \geq 0 \\ (X, 0) & \text{if } X \geq 0 > Y \\ \left(\frac{X+Y}{2}, \frac{X+Y}{2}\right) & \text{if } Y > X > -Y \\ (0, 0) & \text{o/w} \end{cases}$$

(b) Let $$\Lambda(X,Y)^{-1} = \frac{\sup_{\mu_1 \geq \mu_2 \geq 0} L(\mu_1, \mu_2; X, Y)}{L(0,0; X, Y)} = \frac{L(\hat{\mu}_1, \hat{\mu}_2; X, Y)}{L(0,0; X, Y)}$$

Then the LRT statistic is:

$$-2\log\Lambda = -(X-\hat{\mu}_1)^2 - (Y-\hat{\mu}_2)^2 + X^2 + Y^2$$

$$\therefore -2\log\Lambda = \begin{cases} +X^2 + Y^2 & \text{if } X \geq Y \geq 0 \\ +X^2 & \text{if } X \geq 0 > Y \\ +\frac{1}{2}(X+Y)^2 & \text{if } Y > X > -Y \\ 0 & \text{if o/w} \end{cases}$$

Therefore, under the null that $\mu_1 = \mu_2 = 0$,

$$-2\log\Lambda \overset{d}{\approx} \begin{cases} -\chi^2_2 & \text{w.p. } 1/8 \\ -\chi^2_1 & \text{w.p. } 1/4 \\ \frac{1}{2}(Z_1+Z_2)^2 \mid Z_1 > Z_2 > -Z_1 \text{ where } Z_1, Z_2 \overset{iid}{\sim} N(0,1) & \text{w.p. } 1/4. \\ ~~\chi^2_2~~\ 0 & \text{w.p. } 3/8 \end{cases}$$

KKT approach:

$$\mathcal{L}(\mu_1,\mu_2) = (X-\mu_1)^2 + (Y-\mu_2)^2 - \lambda(\mu_1-\mu_2) - \eta\mu_2$$

CS - $\hat{\lambda}(\hat{\mu}_1-\hat{\mu}_2)=0$, $\hat{\eta}\hat{\mu}_2=0$

DF - $\hat{\lambda}, \hat{\eta} \geq 0$

~~Maximizing~~ Minimizing $\mathcal{L}$ in (μ_1,μ_2):

$$\hat{\mu}_1 = X + \frac{\lambda}{2} \qquad \hat{\mu}_2 = Y - \frac{\lambda}{2} - \frac{\eta}{2}$$

Case 1: $\lambda=0, \eta=0 \Rightarrow (\hat{\mu}_1,\hat{\mu}_2) = (X,Y)$ (feasible iff $X \geq Y \geq 0$)

Case 2: $\lambda=0, \eta>0 \Rightarrow \hat{\mu}_2 = 0$ by CS $\Rightarrow \eta = -2Y \Rightarrow (\hat{\mu}_1,\hat{\mu}_2) = (X,0)$

(feasible iff $X \geq 0$, ~~$Y>$~~ $Y<0$)

Case 3: $\lambda>0, \eta=0 \Rightarrow \hat{\mu}_1=\hat{\mu}_2$, $\hat{\mu}_2>0 \Rightarrow \lambda = Y-X$

$\Rightarrow \hat{\mu}_1 = \hat{\mu}_2 = \frac{X+Y}{2}$ (feasible iff $Y>X$, $Y>-X$, ~~$Y>|X|$~~ i.e. $Y>|X|$)

Case 4: $\lambda>0, \eta>0 \Rightarrow \hat{\mu}_1 = \hat{\mu}_2 = 0$

2016 Q5

(a) $$L(\theta;x) = \prod_{i=1}^{n} \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}}\left[e^{-\frac{(x_i-\theta)^2}{2}} + e^{-\frac{(x_i+\theta)^2}{2}}\right]$$

$$\propto \prod_{i=1}^{n} e^{-\frac{x_i^2}{2}}\, e^{-\frac{\theta^2}{2}}\left(e^{x_i\theta} + e^{-x_i\theta}\right)$$

$$\therefore\ \ell(\theta;x) = -n\frac{\theta^2}{2} - \sum\frac{x_i^2}{2} + \sum\log\left(e^{|x_i|\theta} + e^{-|x_i|\theta}\right)$$

This is cts and $\to -\infty$ as $\theta \to \pm\infty$ so an optimiser exists

$$\therefore\ \frac{\partial \ell}{\partial\theta} = -n\theta + \sum\frac{|x_i|e^{|x_i|\theta} - |x_i|e^{-|x_i|\theta}}{e^{|x_i|\theta} + e^{-|x_i|\theta}}$$

$\therefore$ $\theta = 0$ is a stationary point and $\ell' \to -\infty$ as $\theta \to \infty$

$$\therefore\ \frac{\partial^2\ell}{\partial\theta^2} = -n + \sum\frac{\left(e^{|x_i|\theta}+e^{-|x_i|\theta}\right)\left(|x_i|^2e^{|x_i|\theta} + |x_i|^2e^{-|x_i|\theta}\right) - \left(|x_i|e^{|x_i|\theta} - |x_i|e^{-|x_i|\theta}\right)^2}{\left(e^{|x_i|\theta}+e^{-|x_i|\theta}\right)^2}$$

$$= -n + \sum\frac{|x_i|^2\left(e^{|x_i|\theta}+e^{-|x_i|\theta}\right)^2 - |x_i|^2\left(e^{|x_i|\theta}-e^{-|x_i|\theta}\right)^2}{\left(e^{|x_i|\theta}+e^{-|x_i|\theta}\right)^2}$$

$$= -n + \sum\frac{4x_i^2}{\left(e^{|x_i|\theta}+e^{-|x_i|\theta}\right)^2}$$

Now the denominator $e^{|x_i|\theta} + e^{-|x_i|\theta}$ is clearly an increasing function of θ, therefore ℓ'' is a decreasing cts. func of θ.

~~Thus the global~~

Therefore, l' either has a unique root at $\theta=0$ (if $l''(0) \leq 0$) or else l' has exactly 2 roots at $\theta=0$ and at some other $\theta>0$.

$$\left(\frac{\partial^3 l}{\partial \theta^3} = \sum -2 \frac{x_i^2 \overset{>0}{(|x_i| e^{|x_i|\theta} - |x_i| e^{-|x_i|\theta})}}{(e^{|x_i|\theta} + e^{-|x_i|\theta})^3} \leq 0 \right)$$

$$\left(\begin{aligned} \frac{\partial^4 l}{\partial \theta^4} &= -2 \sum x_i^2 \frac{1}{(e^{x_i\theta} + e^{-x_i\theta})^6} \left[x_i^2 (e^{|x_i|\theta} + e^{-|x_i|\theta})^4 - 3x_i^2 (e^{x_i\theta} + e^{-x_i\theta})^2 (e^{x_i\theta} - e^{-x_i\theta})^2 \right] \\ &= -2 \sum \frac{x_i^4 (e^{2x_i\theta} + 2 + e^{-2x_i\theta} - 3e^{2x_i\theta} + 6 - 3e^{-2x_i\theta})}{(e^{x_i\theta} + e^{-x_i\theta})^4} \end{aligned} \right)$$

Thus, if $l''(\theta=\varepsilon; x) \leq - l''(\theta=0; x)$,

then $\hat{\theta}_{MLE} < 2\varepsilon$. To see this, note that

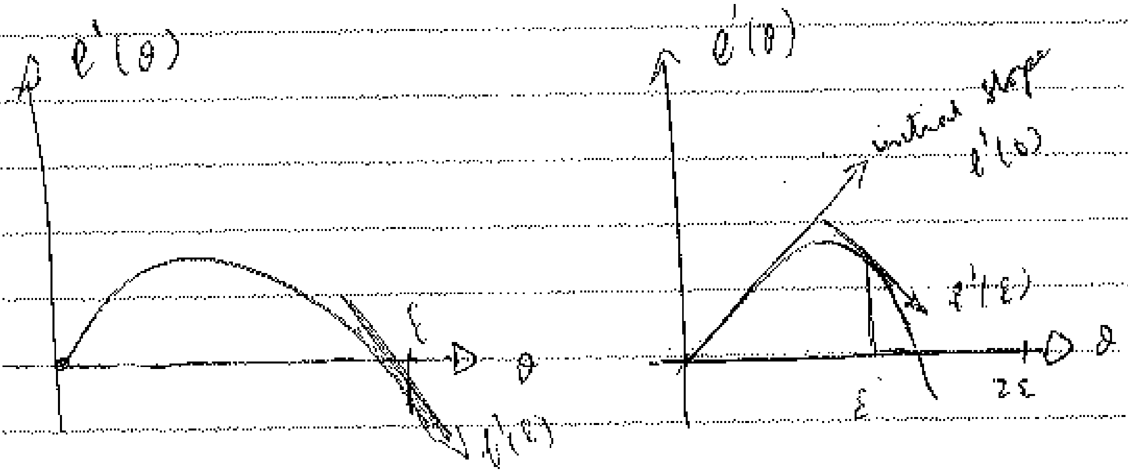
If $l''(\theta=0) \leq 0$, then $\hat{\theta}_{MLE} = 0 < 2\varepsilon$

Else, if $l''(\theta=0) > 0$, then the initial slope of l' is +ve. But by the time the slope has become as −ve as it was +ve at $\theta=0$, as the slope of l'

2016 Q5

is strictly decreasing $\left(\ell''' < 0 \ \cancel{\forall \theta \geq 0}) \ \forall \theta > 0\right)$,

there is at most another ε to cover until we have gone back below 0



Thus,

$$P\left(\ell''(\theta=\varepsilon; x) \leq -\ell''(\theta=0; x)\right) \leq P\left(\hat{\theta}_{MLE} < 2\varepsilon\right)$$

$$LHS = P\left(-n + \sum \frac{4x_i^2}{(e^{|x_i|\varepsilon}+e^{-|x_i|\varepsilon})^2} \leq n - \sum x_i^2\right)$$

$$\cancel{= P\left(n \sum x_i^2 - \sum \frac{4x_i^2}{(e^{x_i\varepsilon}+e^{-x_i\varepsilon})^2} \geq 0\right)}$$

$$\cancel{= P\left(\sum \frac{\left((e^{x_i\varepsilon}+e^{-x_i\varepsilon})^2-4\right)x_i^2}{(e^{x_i\varepsilon}+e^{-x_i\varepsilon})^2} \geq 0\right)}$$

$$\cancel{= P\left(\sum \frac{(e^{x_i\varepsilon}-e^{-x_i\varepsilon})^2 x_i^2}{(e^{x_i\varepsilon}+e^{-x_i\varepsilon})^2} \geq 0\right)}$$

$$= P\left(\sum X_i^2 + \sum \frac{4X_i^2}{(e^{X_i\varepsilon} + e^{-X_i\varepsilon})^2} \le 2n + n \right)$$

$$= P\left(\underbrace{\frac{1}{n}\sum X_i^2}_{\xrightarrow[\text{WLLN}]{P} 1} + \underbrace{\frac{1}{n}\sum \frac{4X_i^2}{(e^{X_i\varepsilon} + e^{-X_i\varepsilon})^2}}_{\xrightarrow{P} 1-\delta \text{ for some } \delta>0 \text{ by WLLN}} \le 1 + 1 \right)$$

$\longrightarrow 1$ as $n \to \infty$ under $\theta_0 = 0$, as $X_i \overset{iid}{\sim} \chi_1^2$

(can easily be shown that $E \underbrace{\frac{4X_i^2}{(e^{X_i\varepsilon} + e^{-X_i\varepsilon})^2}}_{\ge 4 + X_i^2\varepsilon^2} < 1 = E X_i^2$)

Hence $P(\hat{\theta}_{MLE} < 2\varepsilon) \longrightarrow 1 \quad \forall \varepsilon > 0$

so that $\hat{\theta}_{MLE} \xrightarrow{P} 0$. $\Box$

Alternatively, can show $\forall \delta > 0$ fixed

$\underbrace{\frac{1}{n} l_n'(\delta; X)} \xrightarrow{P}$ something < 0

$\overbrace{\delta X_i - \frac{\delta^3 X_i^3}{3}} + \ldots$

$= -\delta + \frac{1}{n}\sum_i^n X_i \tanh(X_i\delta) \xrightarrow{P} -\delta + E_{\mu=0} X \tanh(\delta X_i) < -\delta + \delta E X_i^2 = 0$

$\tanh(x) < x \quad \forall x > 0$

$\therefore E_{\mu=0} X \tanh(\delta X) < E X^2 = 1$

2016 Q5

(b) $P(\hat{\theta}_n = 0) = P(\ell''(\theta=0) \leq 0)$

$$= P\left(n \geq \sum_i^n x_i^2\right)$$

$$= P\left(n \geq \chi_n^2\right)$$

$$= P\left(1 \geq \frac{\chi_n^2}{n}\right)$$

$$= P\left(\sqrt{n}\left(\frac{\sum_i^n x_i^2}{n} - 1\right) \leq 0\right)$$

$$\Rightarrow \longrightarrow \frac{1}{2} \quad \text{under} \quad \theta_0 = 0.$$

(c) By class results, as the MLE is consistent (part a)

and conditions A0 - A4 hold,

it follows that the MLE is asymptotically efficient

Thus,

$$\sqrt{n}\left(\hat{\theta}_{MLE} - 0\right) = \sqrt{n}\,\hat{\theta}_{MLE} \longrightarrow \text{Normal}$$

But we don't have A0 - A4 here ☹.

$$\sqrt{n}\,\hat{\theta}_{MLE}^2 \longrightarrow \begin{cases} 0 & \text{w.p. } 1/2 \\ N(0, \tfrac{1}{2})_+ & \text{w.p. } 1/2 \end{cases}$$

We use a direct argument:

Fix $t \geq 0$.

$$P(n^\alpha \hat{\theta}_n \leq t) \geq P(n^\alpha \hat{\theta}_n = 0) = P(\hat{\theta}_n = 0) \longrightarrow \frac{1}{2}$$

$$\therefore \liminf_{n\to\infty} P(n^\alpha \hat{\theta}_n \leq t) \geq \frac{1}{2}$$

Now to find an upper bound on $P(n^\alpha \hat{\theta}_n \leq t)$, note

$$l''(\theta = t; x) \geq 0 \implies \hat{\theta}_n > t$$

$$\therefore P(\hat{\theta}_n > t) \geq P(l''(\theta = t; x) \geq 0)$$

$$\therefore P(\hat{\theta}_n \leq t) \leq P(l''(\theta = t; X) < 0)$$

$$\therefore P(n^\alpha \hat{\theta}_n \leq t) = P(\hat{\theta}_n \leq t n^{-\alpha}) \leq P\left(-n + \sum \frac{4X_i^2}{(e^{|X_i| t n^{-\alpha}} + e^{-|X_i| t n^{-\alpha}})^2} < 0\right)$$

$$= P\left(\sum \frac{X_i^2}{\cosh(|X_i| t n^{-\alpha})^2} < n\right)$$

$$= P\left(\frac{1}{n} \sum X_i^2 \operatorname{sech}(X_i t n^{-\alpha})^2 < 1\right)$$

$$= P\left(\frac{1}{n} \sum X_i^2 \left(1 - \frac{X_i^2 t^2}{2n^{2\alpha}} + \text{h.o.t.}\} O_p(n^{-4\alpha})\right)^2 < 1\right)$$

$$= P\left(\left(\frac{\sum X_i^2}{n} - 1\right) - \frac{1}{n} \sum \frac{X_i^4 t^2}{n^{2\alpha}} + O_p(n^{-4\alpha}) < 0\right)$$

$$= P\left(\sqrt{n}\left(\frac{\sum X_i^2}{n} - 1\right) < n^{\frac{1}{2} - 2\alpha} t^2 \sum \frac{X_i^4}{n} \} + O_p(n^{-4\alpha})\right)$$

2016 Q5

Now note $\sqrt{n}\left(\frac{\sum X_i^2}{n} - 1\right) \xrightarrow{d} N(0,2)$ by CLT

and $\frac{\sum X_i^4}{n} \xrightarrow{p} 3$ by WLLN

- Therefore, if $\alpha > \frac{1}{4}$, $\frac{1}{2} - 2\alpha < 0$ so that

our bound becomes

$$\limsup_{n\to\infty} P(n^\alpha \hat{\theta}_n \le t) \le \frac{1}{2}$$

- If $\alpha < \frac{1}{4}$, then $\frac{1}{2} - 2\alpha > 0$ and our bound becomes

$$\limsup_{n\to\infty} P(n^\alpha \hat{\theta}_n \le t) \le P(N(0,2) < \infty) = 1 \quad \text{(trivial)}$$

- If $\alpha = \frac{1}{4}$, then our bound becomes

$$\limsup_{n\to\infty} P(n^\alpha \hat{\theta}_n \le t) \le P(N(0,2) < 3t^2) = \Phi\left(\frac{3}{\sqrt{2}} t^2\right)$$

Thus, if $\alpha > \frac{1}{4}$, $n^\alpha \hat{\theta}_n \xrightarrow{d} \begin{cases} 0 & \text{w.p. } 1/2 \\ \infty & \text{w.p. } 1/2 \end{cases}$

On the other hand, following the reasoning from (a), we have

$$P(n^{\alpha}\hat{\vartheta}_n \leq t) = P(\hat{\vartheta}_n \leq n^{-\alpha} t) \geq P\left(\ell''(\vartheta = \tfrac{tn^{-\alpha}}{2}; X) \leq -\ell''(\vartheta = 0, X)\right)$$

$$= P\left(\Sigma X_i^2 + \Sigma X_i^2 \operatorname{sech}(x_i t n^{-\alpha}/2)^2 \leq 2n\right)$$

$$= P\left(\frac{1}{n}\Sigma X_i^2 + \frac{1}{n}\Sigma X_i^2 \left(1 - \frac{X_i^2 t^2}{8n^{2\alpha}} + O_p(n^{-4\alpha})\right)^2 \leq 2\right)$$

$$= P\left(2\left(\frac{\Sigma X_i^2}{n} - 1\right) - \frac{1}{n}\Sigma \frac{X_i^4 t^2}{4n^{2\alpha}} + O_p(n^{-4\alpha}) \leq 0\right)$$

$$= P\left(\frac{\Sigma X_i^2}{n} - 1 \leq n^{-2\alpha} \Sigma \frac{X_i^4 t^2}{8n} + O_p(n^{-4\alpha})\right)$$

$$= P\left(\sqrt{n}\left(\frac{\Sigma X_i^2}{n} - 1\right) \leq n^{\frac{1}{2}-2\alpha} \Sigma \frac{X_i^4 t^2}{8n} + O_p(n^{\frac{1}{2}-4\alpha})\right)$$

Therefore, if $\alpha = \frac{1}{4}$, we have

$$\liminf_{n\to\infty} P(n^{\alpha}\hat{\vartheta}_n \leq t) \geq P(N(0,2) \leq \tfrac{3}{8}t^2) = \Phi\left(\frac{3}{8\sqrt{2}}t^2\right)$$

whereas if $\alpha < \frac{1}{4}$, $\frac{1}{2} - 2\alpha > 0$ so that

$$\liminf_{n\to\infty} P(n^{\alpha}\hat{\vartheta}_n \leq t) \geq 1, \text{ so that } n^{\alpha}\hat{\vartheta}_n \xrightarrow{P} 0$$

Lastly, for $\alpha = \frac{1}{4}$, $\forall t \geq 0$

$$\Phi\left(\frac{3}{8\sqrt{2}}t^2\right) \leq \liminf P(n^{\frac{1}{4}}\hat{\vartheta}_n \leq t) \leq \limsup P(n^{\frac{1}{4}}\hat{\vartheta}_n \leq t) \leq \Phi\left(\frac{3}{\sqrt{2}}t^2\right)$$

2016 Q5

(c) Note the following:

$\ell'(0;X) = 0$

$\ell''(0;X) = \sum X_i^2 - n$

$\ell'''(0;X) = 0$

$\ell^{(4)}(\theta;X) = \sum X_i^4 f(\theta X_i)$

where $f(x) = -2\operatorname{sech}^2(x) + 4\operatorname{sech}^2(x)\tanh^2(x) \longrightarrow -2$ as $x \to 0$.

$\Rightarrow$ f is non-zero on a bounded neighborhood of 0.

Taylor expand:

$0 = \ell'(\hat\theta_n;X) = \hat\theta_n \ell''(0;X) + \frac{1}{6}\hat\theta_n^3 \ell_n^{(4)}(\xi_n|X)$ where $\xi_n \in (0,\hat\theta_n)$

$$\therefore \sqrt{n}\,\hat\theta_n^2 = \frac{-\frac{1}{\sqrt{n}}\ell''(0;X) \xrightarrow{d} N(0,2)}{\frac{1}{6}\frac{1}{n}\ell_n^{(4)}(\xi_n;X)} \quad \text{if } \ell''(0;X) > 0$$

$$= 0 \quad \text{if } \ell''(0;X) < 0$$

Now $\xi_n \xrightarrow{P} 0$ as MLE is consistent

and $\frac{1}{n}\ell^{(4)}(0;X) \xrightarrow{P} -6$, because:

$\sup_{\theta \in [0,\delta]} \left|\frac{1}{n}\ell^{(4)}(\theta;X) - E\,\ell^{(4)}(\theta;X)\right| \xrightarrow{P} 0$ by:

~~ULN~~ ULLN (uniform law of large numbers)

[if $\ell_n^{(4)}(\theta;X)$ is cts in θ $\forall x$, $|\ell^{(4)}(\theta;X)| \le h(X)$, $E\,h(X) < \infty$]

2015 Q1

(a) $f(h) = o(|h|)$ as $h \to 0$

means $\left|\frac{f(h)}{h}\right| \longrightarrow 0$ as $h \to 0$

i.e. $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall h \in (-\delta, \delta)$,

$$\left|\frac{f(h)}{h}\right| < \varepsilon.$$

Now suppose also $X_n \xrightarrow{p} 0$. ~~Then we have that~~

~~Pf~~ We want to show $f(X_n) = o_p(|X_n|)$,

i.e. $\frac{f(X_n)}{|X_n|} \xrightarrow{p} 0$. But using the above,

$$P\left(\frac{f(X_n)}{|X_n|} > \varepsilon\right) \leq P(|X_n| > \delta) \longrightarrow 0 \text{ as } n \to \infty \quad \square$$

(b) $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, \theta^2)$

$$\therefore \ L(\theta; X) = (2\pi\theta^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\theta^2} \Sigma (X_i - \theta)^2\right\}$$

$$= (2\pi\theta^2)^{-\frac{n}{2}} \exp\left\{-\frac{\Sigma X_i^2}{2\theta^2} + \frac{n\bar{X}}{\theta} - \frac{n}{2}\right\}$$

this is a curved exponential family, with

$T_1 = \Sigma X_i^2$, $\eta_1 = -\frac{1}{2\theta^2}$, $T_2 = \bar{X}$, ~~T_2~~ $\eta_2 = \frac{n}{\theta}$

As $\{(\eta_1(\theta), \eta_2(\theta)), \theta \in \mathbb{R}\}$ is a (non-straight) curve in 2D,

(T_1, T_2) is M.S. by class results (can pick ~~two vectors~~

$V_0, V_1, V_2 \in \{ (\eta_1(\theta), \eta_2(\theta)) : \theta \in \mathbb{R} \}$ s.t

$V_1 - V_0$, $V_2 - V_0$ are lin. indep.

However, $E T_1 = \Sigma E X_i^2 = \Sigma \text{Var}\, x_i + E^2 x_i = 2n\theta^2$

~~$E T_2 = E X_1 = \theta$~~ $T_2 \sim N(\theta, \frac{\theta^2}{n})$

∴ $E T_2^2 = \frac{\theta^2}{n} + \theta^2 = \frac{n+1}{n} \theta^2$

∴ $\frac{T_1}{2n} - \frac{n}{n+1} T_2^2$ is a ~~function~~ non-zero function

of the MS statistic with ~~constant~~ expectation 0 $\forall \theta$.

∴ (T_1, T_2) is not complete, so no ~~comp~~ CS. statistic exists.

(c) $\beta(\theta) \le 1$ as ϕ is a test fun.

∴ $\beta(\theta)$ is flat (convex and bounded)

∴ $E_\theta \phi(X) = \alpha$ $\forall \theta$

2015 Q1

(d) $$E_i \beta_\phi(\theta) = \int \beta_\phi(\theta)\, \pi_i(\theta)\, d\theta$$

$$= \int \left\{ \int \phi(x)\, p_\theta(x)\, dx \right\} \pi_i(\theta)\, d\theta$$

$$= \int\int \phi(x)\, p_\theta(x)\, \pi_i(\theta)\, d\theta\, dx \qquad \text{(Fubini)}$$

$$= \int \phi(x) \left\{ \int p_\theta(x)\, \pi_i(\theta)\, d\theta \right\} dx$$

$$= \int \phi(x)\, m_i(x)\, dx$$

So assuming p_θ and π_i have densities,

it suffices to find the ~~MP test~~ level α MP test

for $X \sim m_0$ vs $X \sim m_1$, where m_i denotes

the marginal distn. of X when θ follows the π_i prior.

∴ the optimal test is the NP test

$$\phi^*(X) = 1 \quad \text{if} \quad m_1(X) > k\, m_0(X)$$
$$= 0 \quad \text{if} \quad m_1(X) < k\, m_0(X)$$

~~st~~ where k is s.t. $E_m \int \phi(x)\, m_0(x)\, dx = \alpha$.

(e) By definition, the RV Z that minimizes $E(Z-T)^2$ is the projection of T into S. By class results, it suffices to check that $Z = \Sigma E(T|X_i)$ satisfies

$$E\left(T - \Sigma E(T|X_i)\right) \cdot Y = 0 \quad \forall Y \in S.$$

Let $Y = \sum_i^n g_i(X_i)$. Then

$$E\,TY = \Sigma E\,T g_i(X_i) = \Sigma E\left[E\left(T g_i(X_i) | X_i\right)\right] =$$
$$= \Sigma E\left[g_i(X_i) E[T|X_i]\right] = E\left[\Sigma g_i(X_i) E(T|X_i)\right]$$

$$E\,Y \Sigma E(T|X_i) = E\left[\left(\Sigma g_i(X_i)\right)\left(\Sigma E(T|X_i)\right)\right] =$$
$$= E\left[\Sigma g_i(X_i) E(T|X_i)\right] + E\left[\sum_{i \neq j} E(g_i(X)) E(T|X_i)\right]$$
$$= E\left[\Sigma g_i(X_i) E(T|X_i)\right] + E(g_i(X)) \underbrace{E\left[E(T|X_i)\right]}_{= ET = 0}$$
$$= E\left[\Sigma g_i(X_i) E(T|X_i)\right]$$

$\therefore$ $E\,TY = E\,ZY$ $\therefore$ $E(T-Z)Y = 0 \quad \forall Y \in$ ~~B~~ S

$\therefore$ Z is a projection $\square$.

2015 Q2

(a) $p_\theta(\vec{x}) = p_\theta(x_n \mid x_1, \dots, x_{n-1})\, p(x_{n-1} \mid x_1, \dots, x_{n-2}) \cdots p(x_2 \mid x_1)\, p(x_1)$

$$= \theta^{\mathbb{1}\{x_n = x_{n-1}\}} (1-\theta)^{\mathbb{1}\{x_n \neq x_{n-1}\}} \cdots \theta^{\mathbb{1}\{x_2 = x_1\}} (1-\theta)^{\mathbb{1}\{x_2 \neq x_1\}} \cdot \frac{1}{2}$$

$$= \theta^{\sum_{i=2}^{n} \mathbb{1}\{x_i = x_{i-1}\}} (1-\theta)^{\sum_{2}^{n} \mathbb{1}\{x_i \neq x_{i-1}\}} \cdot \frac{1}{2}$$

Now note that $\mathbb{1}\{x_i = x_{i-1}\} = \frac{1}{2}(x_i x_{i-1} + 1)$

(as $x_i \in \{-1, 1\}$) $\mathbb{1}\{x_i \neq x_{i-1}\} = 1 - \mathbb{1}\{x_i = x_{i-1}\} = \frac{1}{2}(1 - x_i x_{i-1})$

$$\therefore\ p_\theta(\vec{x}) = \theta^{\sum_{2}^{n} \frac{1}{2}(x_i x_{i-1} + 1)} (1-\theta)^{\sum_{2}^{n} \frac{1}{2}(1 - x_i x_{i-1})} \frac{1}{2}$$

$$= \frac{1}{2}\, \theta^{\frac{n-1}{2}}\, \theta^{\frac{1}{2}\sum_{1}^{n-1} x_i x_{i+1}} (1-\theta)^{+\frac{n-1}{2}} (1-\theta)^{-\frac{1}{2}\sum_{1}^{n-1} x_i x_{i+1}}$$

$$= \frac{1}{2} \exp\left\{ \frac{1}{2} T \log\theta - \frac{1}{2} T \log(1-\theta) + \frac{n-1}{2} \log(\theta(1-\theta)) \right\}$$

$$= \frac{1}{2} \exp\left\{ \frac{1}{2} T \log\frac{\theta}{1-\theta} + \frac{n-1}{2} \log(\theta(1-\theta)) \right\}$$

This is a 1-parameter exponential family with

$$T = \sum_{1}^{n-1} x_i x_{i+1}, \quad \eta(\theta) = \frac{1}{2}\log\frac{\theta}{1-\theta}, \quad B(\theta) = \frac{n-1}{2}\log(\theta(1-\theta))$$

As $\theta \in [\frac{1}{2}, 1)$, $\eta(\theta) \in [0, \infty)$ which has non-empty interior. By class results, T is sufficient (and M.S. and C.S.)

(b) Our test is equivalent to $H_0: \eta = 0$ vs $H_1: \eta > 0$.

Since $p_\eta(\vec{x}) = \frac{1}{2}\exp\left\{ T\eta - A(\eta) \right\}$, our family is

MLR in T so, by class results, $\exists$ a UMP ϕ of the form

(I) $\phi(X) = 1$ if $T > c$

$= \nu$ if $T = c$

$= 0$ if $T < c$

s.t. $E_{\theta=0}\,\phi(X) = \alpha$.

Imposing the level constraint gives

$$P_{\theta=\frac{1}{2}}(T > c) + \nu P_{\theta=\frac{1}{2}}(T = c) = \alpha$$

∴ ~~P (Bin~~ But under ~~θ=0~~, $\theta = \frac{1}{2}$, $T \overset{d}{=} 2\,\text{Bin}(n, \tfrac{1}{2}) - n$

$$\therefore\ P\left(\text{Bin}(n,\tfrac{1}{2}) > \frac{c+n}{2}\right) + \nu P\left(\text{Bin}(n,\tfrac{1}{2}) = \frac{c+n}{2}\right) = \alpha$$

∴ $\frac{c+n}{2}$ is the unique integer k_0 s.t.

$$P(\text{Bin}(n,\tfrac{1}{2}) > k_0) < \alpha, \quad P(\text{Bin}(n,\tfrac{1}{2}) \geq k_0) \geq \alpha$$

and $$\nu = \frac{\alpha - P(\text{Bin}(n,\frac{1}{2}) > k_0)}{P(\text{Bin}(n,\frac{1}{2}) = k_0)}$$

This fully specifies I □

n-1 n-1

(c) Under $H_0: \theta = \frac{1}{2}$, $T \overset{d}{=} 2\,\text{Bin}(n, \tfrac{1}{2}) - n$

$$\therefore\ ET = 2(n\tfrac{1}{2}) - n = 0$$

$$\text{Var}\,T = 4\text{Var}\{\text{Bin}(n,\tfrac{1}{2})\} = n$$

2015

(d) As discussed, $T \overset{d}{=} 2\,\mathrm{Bin}(n, \frac{1}{2}) - n$,

so $P(T=t) = P\left(\mathrm{Bin}(n,\frac{1}{2}) = \frac{t+n}{2}\right)$

$$= \underbrace{\binom{n}{\frac{t+n}{2}} \cdot \left(\frac{1}{2}\right)^n} \quad \text{under } H_0$$

for t s.t. $\frac{t+n}{2} \in \{0, \dots, n\}$,

i.e. $t \in \{0, 2, \dots, 2n\} - n$ □

(e) Assume $T \overset{d}{\approx} N(0, n)$, then our test I

has level ~~condition~~ constraint

$$E_{\eta=0}\, \phi(X) = \alpha \implies P(T > c) = \alpha$$

$$\therefore \quad P\left(N(0,1) > \frac{c}{\sqrt{n}}\right) = \alpha$$

$$\therefore \quad \frac{c}{\sqrt{n}} = z_{1-\alpha}$$

$$\therefore \quad c = z_{0.95} \cdot \sqrt{40}$$

$\therefore$ Reject if $T > z_{0.95}\sqrt{40}$.

(f) We are in the setup of part e, with exactly n = 40. Here,

$$T = \underbrace{2-2+3-1}_{1^{st}\text{ quarter}}+\underbrace{5-2+2-2+1-1}_{2^{nd}\text{ quarter}}+\underbrace{1-2+5-1}_{3^{rd}\text{ quarter}}+\underbrace{2-2+3-2}_{4^{th}\text{ quarter}}$$

$= 9$

Recall our UMP from the previous part reject if this value is greater than $z_{0.95} \cdot \sqrt{40} = 2 \cdot z_{0.95} \cdot \sqrt{10} \simeq 2 \cdot 2 \cdot 3$
$\simeq 12$

$\therefore$ We do not have evidence to reject the null ~~[illegible]~~ at level 5% ☐

~~2015~~
2015 Q3

(a) $L(\theta; X) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} (X-\theta)^2 \right\}$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}X^2 + X\theta - \frac{1}{2}\theta^2 \right\}.$$

~~ETX~~ $|X|$ is not sufficient as the likelihood does not factorize in the required form (N-F factorization criterion).

More explicitly, note that

$$P_{\theta=2}(X=2 \mid |X|=2) = \frac{f_{\theta=2}(2)}{f_{\theta=2}(2) + f_{\theta=2}(-2)} = \frac{1}{1+e^{-8}}$$

$$P_{\theta=-2}(X=2 \mid |X|=2) = \frac{f_{\theta=-2}(2)}{f_{\theta=-2}(2) + f_{\theta=-2}(-2)} = \frac{e^{-8}}{1+e^{-8}}$$

$\therefore$ the distn. of $(X \mid |X|)$ is NOT free of θ ◼.

(b) $\ell(\theta; X) = \text{constant} - \frac{1}{2}\theta^2 + X\theta = -\frac{1}{2}(\theta - X)^2 + \text{constant}$

~~this is maximal at~~ this is maximized at the minimal value of $(\theta - X)^2$, so we pick $\hat{\theta}_{ML} = 2\,\text{sign}(X)$ ☐

(c) $\pi(\theta) = \begin{cases} 1/2 & \text{if } \theta = 2 \\ 1/2 & \text{if } \theta = -2 \end{cases}$

$R(\theta, \delta(X)) =$ ~~$E_\theta L(\theta, \hat{\theta}) = P_\theta$~~ $E_\theta L(\theta, \delta(X)) = P_\theta(\delta(X) \neq \theta)$

$\therefore\ r(\pi, \delta) = E_{\theta \sim \pi} R(\theta, \delta) = \frac{1}{2} P_{\theta=2}(\delta(X) \neq 2) + \frac{1}{2} P_{\theta=-2}(\delta(X) \neq -2)$

As we have 0-1 loss, the Bayes estimator is the posterior mode.

$\pi(\theta|x) \propto L(\theta|x)\pi(\theta)$ ~~$\propto$~~

$$\therefore\ \pi(\theta|x) \propto \begin{cases} e^{-\frac{1}{2}(x-2)^2} & \text{if } \theta = 2 \\ e^{-\frac{1}{2}(x+2)^2} & \text{if } \theta = -2 \end{cases}$$

$$\therefore\ \pi(\theta|x) = \begin{cases} \dfrac{e^{-\frac{1}{2}(x-2)^2}}{e^{-\frac{1}{2}(x-2)^2} + e^{-\frac{1}{2}(x+2)^2}} & \text{if } \theta = 2 \\ \dfrac{e^{-\frac{1}{2}(x+2)^2}}{e^{-\frac{1}{2}(x-2)^2} + e^{-\frac{1}{2}(x+2)^2}} & \text{if } \theta = -2 \end{cases}$$

$\therefore$ ~~the Bayes estimator is~~

~~$\delta_\pi(x) = 2\mathbb{1}\{$~~

$$= \begin{cases} \dfrac{e^{2x}}{e^{2x} + e^{-2x}} & \text{if } \theta = 2 \\ \dfrac{e^{-2x}}{e^{2x} + e^{-2x}} & \text{if } \theta = -2 \end{cases}$$

$\therefore$ the Bayes estimator is

$$\delta_\pi(x) = 2\,\mathbb{1}\{e^{2x} > e^{-2x}\} + (-2)\,\mathbb{1}\{e^{2x} < e^{2x}\}$$

$$= 2\,\mathbb{1}\{x > 0\} + (-2)\,\mathbb{1}\{x < 0\}$$

$$= 2 - 4\,\mathbb{1}\{x < 0\} \qquad \square$$

2015 Q3

(c) We compute the risk of our previous Bayes estimator.

$$R(\theta,\delta_\pi) = E_\theta L(\theta,\delta_\pi(X)) = P_\theta(\delta_\pi(X) \neq \theta)$$

$$= P_\theta(2 - 4\mathbb{1}_{\{X<0\}} \neq \theta)$$

$$= \begin{cases} P(N(2,1)<0) & \text{if } \theta=2 \\ P(N(-2,1)>0) & \text{if } \theta=-2 \end{cases}$$

$$= \begin{cases} \Phi(-2) & \text{if } \theta=2 \\ \Phi(-2) & \text{if } \theta=-2 \end{cases}$$

$\therefore$ δ_π is a Bayes estimator with constant risk.

$\therefore$ δ_π is minimax $\square$

2015 Q4

(a) $E\,(X-\theta)g(X) = \int_{-\infty}^{\infty} (x-\theta)\,g(x)\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(x-\theta)^2}\,dx$

integrate by parts

$$\downarrow$$

$$= -\left[\sigma^2 g(x)\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(x-\theta)^2}\right]_{-\infty}^{\infty} + \sigma^2\int_{-\infty}^{\infty} g'(x)\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(x-\theta)^2}\,dx$$

$$= \sigma^2 E\, g'(X)$$

(writing $\frac{d}{dx}\left\{e^{-\frac{1}{2\sigma^2}(x-\theta)^2}\right\} = -\frac{(x-\theta)}{\sigma^2}e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$) $\square$

(b) $\pi(\theta|x) \propto L(\theta;x)\,\gamma(\theta) \propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2}\,\gamma(\theta)$

$$\therefore\ \pi(\theta|x) = \frac{\exp\{-\frac{1}{2\sigma^2}(x-\theta)^2\}\,\gamma(\theta)}{\int \exp\{-\frac{1}{2\sigma^2}(x-\theta)^2\}\,\gamma(\theta)\,d\theta} = \frac{p_\theta(x)\,\gamma(\theta)}{f(x)}$$

$$\therefore\ E[\theta|x] = \int \theta\,\pi(\theta|x)\,d\theta$$

$$= \frac{\int \theta \exp\{-\frac{1}{2\sigma^2}(x-\theta)^2\}\,\gamma(\theta)\,d\theta}{\int \exp\{-\frac{1}{2\sigma^2}(x-\theta)^2\}\,\gamma(\theta)\,d\theta}$$

$$= \frac{\int (\theta-x)\exp\{-\frac{1}{2\sigma^2}(x-\theta)^2\}\,\gamma(\theta)\,d\theta}{\int \exp\{-\frac{1}{2\sigma^2}(X-\theta)^2\}\,\gamma(\theta)\,d\theta} + X\ \cancelto{1}{\frac{\int \cdots\,d\theta}{\int -\,d\theta}}$$

$$= \frac{E_{\theta\sim N(x,\sigma^2)}\left[(\theta-x)\,\gamma(\theta)\right]}{f(X)} + X$$ (times and divide by $\sqrt{2\pi\sigma^2}$)

$$= \frac{\sigma^2\,E_{\theta\sim N(x,\sigma^2)}\,\gamma'(\theta)}{f(X)} + X$$ (by part (a))

$$\overset{\textcircled{I}}{=} X + \sigma^2 \frac{f'(X)}{f(X)}$$ as required,

~~since~~ where I follows because:

$$f'(X) = \frac{d}{dX} \int \ell \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X-\theta)^2}{2\sigma^2}} \gamma(\theta)\,d\theta$$

$$\cancel{= \int \frac{1}{\sqrt{2\pi\sigma^2}} \frac{d}{dX} e^{-\frac{(X-\theta)^2}{2\sigma^2}} \gamma(\theta)\,d\theta}$$

$$\cancel{= \int \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \left(\frac{-(X-\theta)}{\sigma^2}\right) e^{-\frac{(X-\theta)^2}{2\sigma^2}} \gamma(\theta)\,d\theta}$$

$$= \frac{d}{dX} \int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\tilde{\theta}^2}{2\sigma^2}} \gamma(X-\tilde{\theta})\,d\tilde{\theta} \qquad (\tilde{\theta} = X-\theta)$$

$$= \int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\tilde{\theta}^2}{2\sigma^2}} \frac{d}{dX} \gamma(X-\tilde{\theta})\,d\tilde{\theta}$$

$$= \int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\tilde{\theta}^2}{2\sigma^2}} \gamma'(X-\tilde{\theta})\,d\tilde{\theta}$$

$$= \int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X-\theta)^2}{2\sigma^2}} \gamma'(\theta)\,d\theta \qquad (\theta = X-\tilde{\theta})$$

$$= E_{\theta\sim N(X,\sigma^2)}\left[\gamma'(\theta)\right] \qquad \square$$

2015 Q4

(c) $B(\gamma) = E\,\theta^2 - 2\,E\left[\theta\,E[\theta|X]\right] + E\left[E(\theta|X)^2\right]$

$$= E\,\theta^2 - 2\,E\,\theta\left(X + \sigma^2 \frac{f'(X)}{f(X)}\right) + E\left[\left(X + \sigma^2 \frac{f'(X)}{f(X)}\right)^2\right]$$

$$= E\,\theta^2 - 2\,E\,\theta X - 2\sigma^2 E\,\theta \frac{f'(X)}{f(X)} + E\,X^2 + 2\sigma^2 E\,X \frac{f'(X)}{f(X)}$$

$$+ \sigma^4 E \frac{f'(X)^2}{f(X)^2}$$

II $$= E\left(X^2 - 2\theta X - \theta^2\right) + 2\sigma^2 E\,(X-\theta)\frac{f'(X)}{f(X)} + \sigma^4 E \frac{f'(X)^2}{f(X)^2}$$

~~$= E\,(X-\theta)^2 + 2\sigma^2\left[\sigma^2 E\right.$~~

~~$= E\,(X-\theta)^2 + 2\sigma^2$~~

Now note • $E\left(X^2 - 2\theta X + \theta^2\right) = E\,(X-\theta)^2$

$$= E\left[E\,(X-\theta)^2|\theta\right] = E\,\sigma^2 = \sigma^2$$

~~$= E\,(X-\theta)^2$~~

• $$E(X-\theta)\frac{f'(X)}{f(X)} = E\left[E\left[(X-\theta)\frac{f'(X)}{f(X)}\,\Big|\,\theta\right]\right]$$

$$= \sigma^2 E\left[E\left[\frac{d}{dX}\left(\frac{f'(X)}{f(X)}\right)\Big|\,\theta\right]\right] \quad \text{(by part a)}$$

$$= \sigma^2 E\left[E\left[\frac{f(X)f''(X) - f'(X)^2}{(f(X))^2}\,\Big|\,\theta\right]\right]$$

$$= \sigma^2 E\left[E\left[\frac{f''(X)}{f(X)}\Big|\,\theta\right] - E\left[\frac{f'(X)^2}{f(X)^2}\Big|\,\theta\right]\right]$$

$$= \sigma^2 E\left[\frac{f''(X)}{f(X)}\right] - \sigma^2 E\left[\frac{f'(X)^2}{f(X)^2}\right]$$

Plugging this back into II ,

$$B(\gamma) = \sigma^2 + 2\sigma^4 E \frac{f''(X)}{f(X)} - \sigma^4 E \frac{f'(X)^2}{f(X)^2}$$

But, ~~E~~ noting that $f(x)$ is the marginal of X,

$$E \frac{f''(X)}{f(X)} = \int \frac{f''(x)}{f(x)} f(x)\, dx = \frac{d^2}{dx^2} \int f(x)\, dx = 0$$

$$E \frac{f'(X)^2}{f(X)^2} = \int \frac{f'(x)^2}{f(x)}\, dx = I(f)$$

$$\therefore \quad B(\gamma) = \sigma^2 (1 - \sigma^2 I(f))$$

~~(d) let $X \sim N(\theta, \sigma^2)$ and $\theta \sim \gamma(\theta)$.~~

~~Then $\sigma^2 Var(\gamma) = Var(X) \cdot Var(\theta)$.~~

~~By Cauchy-Schwarz~~

~~$\sigma^2 Var(\gamma) \geq Cov(X, \theta)^2 = (E X\theta - EX E\theta)^2$~~

~~Now $(EX\theta - EXE\theta)^2 = B(\gamma)(\sigma^2 + Var(\gamma))$~~

~~ie $(EX\theta - EXE\theta)^2 = \left[E(\theta - E(\theta|X))^2\right]\left(E\left[(X-\theta)^2 | \theta\right] + E(\theta - E(\theta))^2\right)$~~

2015 Q4

(d) Note the following:

$$B(\gamma) \leq \frac{\sigma^2 \operatorname{Var}(\gamma)}{\sigma^2 + \operatorname{Var}(\gamma)}$$

$$\iff 1 - \sigma^2 I(f) \leq \frac{\operatorname{Var}(\gamma)}{\sigma^2 + \operatorname{Var}(\gamma)} \qquad (\text{by } (c))$$

$$\iff 1 - \sigma^2 I(f) \leq 1 - \frac{\sigma^2}{\sigma^2 + \operatorname{Var}(\gamma)}$$

$$\iff -I(f) \leq -\frac{1}{\sigma^2 + \operatorname{Var}(\gamma)}$$

$$\iff I(f)\,(\sigma^2 + \operatorname{Var}(\gamma)) \geq 1 .$$

Now note that $\sigma^2 + \operatorname{Var}(\gamma) = \operatorname{Var}(X|\theta) + \operatorname{Var}(\theta)$

$$= E \operatorname{Var}(X|\theta) + \operatorname{Var} E(X|\theta) = \operatorname{Var} X$$

and $I(f) = E_{X \sim f(x)} \dfrac{f'(X)^2}{f(X)^2}$

∴ this reminds us of the information in X and variance in X

marginally, so reminds us of CRLB.

By Cauchy-Schwarz,

$$\operatorname{Var}\left(\frac{f'(X)}{f(X)}\right) \operatorname{Var}(X) \geq \operatorname{Cov}\left(\frac{f'(X)}{f(X)}, X\right)^2$$

where expectations are taken w.r.t. the marginal $f(x)$.

exchanging derivation and integral

∴ But

$$\operatorname{Var}\frac{f'(X)}{f(X)} = E\left(\frac{f'(X)}{f(X)}\right)^2 - \overbrace{E^2\frac{f'(X)}{f(X)}}^{0} = I(f)$$

$$\operatorname{Cov}\left(\frac{f'(X)}{f(X)}, X\right) = E\,\frac{f'(X)}{f(X)}X - \underbrace{E\,\frac{f'(X)}{f(X)}}_{0}\,EX$$

$$= \int \frac{f'(x)}{f(x)}\,x\,f(x)\,dx$$

$$= \int x f'(x)\,dx$$

$$= \Big[x f(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\,dx$$

$$= -1$$

$$\therefore \quad I(f)\,(\sigma^2 + \operatorname{Var}(Y)) \geq 1 \qquad \square$$

2015 Q5

(a) $L(\lambda,\gamma,\eta;y_i,z_i) = \prod_{i=1}^{n} \lambda e^{\gamma z_i} \exp\{-\lambda e^{\gamma z_i} y_i\} \eta^{z_i} (1-\eta)^{1-z_i}$

$\therefore \ \ell(\lambda,\gamma,\eta;y_i,z_i) = \sum_{i=1}^{n}\{\log\lambda + \gamma z_i - \lambda e^{\gamma z_i} y_i + z_i\log\eta + (1-z_i)\log(1-\eta)\}$

$\cancel{= \sum_{i=1}^{n}}$

$$= n\log\lambda + \gamma\sum z_i - \sum \lambda e^{\gamma z_i} y_i + \log\left(\frac{\eta}{1-\eta}\right)\sum z_i + n\log(1-\eta)$$

(~~We see there is no hope of solving the likelihood eqn.~~)

so we check A0 - A4 and that ℓ has a unique maximiser.

$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum e^{\gamma z_i} y_i$, $\frac{\partial \ell}{\partial \gamma} = \sum z_i - \lambda\sum z_i y_i e^{\gamma z_i}$, $\frac{\partial \ell}{\partial \eta} = \frac{\sum z_i}{\eta} - \frac{n-\sum z_i}{1-\eta}$

$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n}{\lambda^2}$, $\frac{\partial^2 \ell}{\partial \gamma^2} = -\lambda\sum z_i^2 y_i e^{\gamma z_i}$, $\frac{\partial^2 \ell}{\partial \eta^2} = -\frac{\sum z_i}{\eta^2} - \frac{n-\sum z_i}{(1-\eta)^2}$

$\frac{\partial^2 \ell}{\partial \lambda \partial \gamma} = -\sum z_i y_i e^{\gamma z_i}$, $\frac{\partial^2 \ell}{\partial \lambda \partial \eta} = 0$, $\frac{\partial^2 \ell}{\partial \gamma \partial \eta} = 0$

$\therefore$ the Hessian matrix is

$$H = \begin{pmatrix} -\frac{n}{\lambda^2} & -\sum z_i y_i e^{\gamma z_i} & 0 \\ -\sum z_i y_i e^{\gamma z_i} & -\lambda\sum z_i^2 y_i e^{\gamma z_i} & 0 \\ 0 & 0 & -(\cdots) \end{pmatrix}$$

To show this is -ve definite, it suffices to check

$$(v_1 \ \ v_2)\begin{pmatrix} -\frac{n}{\lambda^2}v_1 & -v_2\sum z_i y_i e^{\gamma z_i} \\ -v_1\sum z_i y_i e^{\gamma z_i} & -\lambda v_2 \sum z_i^2 y_i e^{\gamma z_i} \end{pmatrix}$$

$$= -\frac{n}{\lambda^2}v_1^2 - 2v_1v_2\sum z_i y_i e^{\gamma z_i} - \lambda v_2^2 \sum z_i^2 y_i e^{\gamma z_i}$$

II

Solving $\frac{\partial \ell}{\partial \gamma} = 0$ gives $\sum z_i = \lambda e^{\gamma} \sum z_i y_i$ (III)

and solving $\frac{\partial \ell}{\partial \lambda} = 0$ gives $\lambda = \frac{n}{\sum y_i e^{\gamma z_i}} = \frac{n}{(e^{\gamma} \sum y_i z_i) + \sum y_i (1-z_i)}$

subbing into III,

$$\sum z_i = \frac{n e^{\gamma}}{e^{\gamma} \sum y_i z_i + \sum y_i - \sum y_i z_i} \sum y_i z_i$$

$$\Rightarrow \quad e^{\gamma} \left(\sum y_i z_i \right) \sum z_i + (\sum z_i)(\sum y_i) - (\sum z_i)(\sum y_i z_i) = n e^{\gamma} \sum y_i z_i$$

$$\Rightarrow \quad e^{\gamma} (\sum y_i z_i)(n - \sum z_i) = (\sum z_i)(\sum y_i - \sum y_i z_i)$$

$$\Rightarrow \quad e^{\gamma} = \frac{(\sum z_i)(\sum y_i - \sum y_i z_i)}{(\sum y_i z_i)(n - \sum z_i)}$$

$$\therefore \hat{\lambda} = \frac{n}{\frac{n(\sum y_i (1-z_i))}{n - \sum z_i}} = \frac{n - \sum z_i}{\sum y_i - \sum y_i z_i}$$

$$= \frac{\sum y_i (1-z_i)}{(\sum y_i z_i)\left(\frac{n}{\sum z_i} - 1 \right)}$$

this is the unique root of the likelihood eqn, and we will check the Hessian is -ve definite at this point

Now note $\frac{\sum z_i}{n} \xrightarrow{p} \eta$ by WLLN so $\frac{n}{\sum z_i} - 1 \xrightarrow{p} \frac{1}{\eta} - 1$

On the other hand, $E Y_i z_i = E\left[E\left[Y_i z_i \mid z_i \right] \right]$

$$= E\left[z_i \frac{1}{\lambda e^{\gamma z_i}} \right] = \frac{\eta}{\lambda} e^{-\gamma} \quad \text{(III)}$$

$$E\, Y_i (1-z_i) = E\left[(1-z_i) \frac{1}{\lambda e^{\gamma z_i}} \right] = \frac{1-\eta}{\lambda}, \quad \text{(IV)}$$

$$E\, Y_i^2 (1-z_i)^2 = E\left[(1-z_i)^2 E\left[Y_i^2 \mid z_i \right] \right] = E\,(1-z_i)^2 \left(\frac{2}{\lambda^2 e^{2\gamma z_i}} \right) = \frac{2(1-\eta)}{\lambda^2}$$

2015 Q5

$$\therefore \ \mathrm{Var}\ Y_i(1-Z_i) = \frac{2(1-\eta)}{\lambda^2} - \frac{(1-\eta)^2}{\lambda^2} = \frac{2-2\eta-1+2\eta-\eta^2}{\lambda^2} = \frac{1-\eta^2}{\lambda^2}$$

$$\therefore \ \sqrt{n}\left(\frac{\sum Y_i(1-Z_i)}{n} - \frac{1-\eta}{\lambda}\right) \xrightarrow{d} N\left(0, \frac{1-\eta^2}{\lambda^2}\right) \quad (CLT)$$

$$\frac{\sum Y_i Z_i}{n} \xrightarrow{p} \frac{\eta}{\lambda} e^{-\gamma} \quad (WLLN)$$

$$\therefore \ \cancel{\sqrt{n}\left(e^{\hat{\gamma}} - \frac{\frac{1-\eta}{\lambda}}{\frac{\eta e^{-\gamma}}{\lambda}\left(\frac{1}{\eta}-1\right)}\right) \xrightarrow{d} N\left(0, \frac{1-\eta^2}{\lambda^2}\right)}$$

$$\& \quad \frac{\sum Y_i Z_i}{n}\left(\frac{n}{\sum Z_i}-1\right) \longrightarrow \frac{\eta}{\lambda} e^{-\gamma}\left(\frac{1}{\eta}-1\right) = \frac{e^{-\gamma}(1-\eta)}{\lambda}$$

$$\therefore \ \frac{\sqrt{n}\left(\frac{\sum y_i(1-Z_i)}{n} - \frac{1-\eta}{\lambda}\right)}{\frac{\sum Y_i Z_i}{n}\left(\frac{n}{\sum Z_i}-1\right)} \xrightarrow{d} \frac{N\left(0, \frac{1-\eta^2}{\lambda^2}\right)}{\frac{e^{-\gamma}(1-\eta)}{\lambda}} = N\left(0, \frac{(1-\eta)}{(1-\eta)} e^{2\gamma}\right)$$

by Slutsky's. $= \sqrt{n}\left(\frac{\frac{\sum y_i(1-Z_i)}{n} - e^{\gamma}\left(\frac{n}{\sum Z_i}-1\right)\frac{\sum y_i Z_i}{n}}{\frac{\sum y_i Z_i}{n}\left(\frac{n}{\sum Z_i}-1\right)}\right) \xrightarrow[\textcircled{*}]{d} N\left(0, e^{2\gamma}\frac{1}{\eta(1-\eta)}\right)$

$\textcircled{*}$ $\therefore \ \sqrt{n}\left(e^{\hat{\gamma}} - e^{\gamma}\right) \cancel{\xrightarrow{d} N\left(0, \frac{\eta(1+\eta)}{(1-\eta)} e^{2\gamma}\right)}$ again by Slutsky's.

→ proof at end.

By Δ-method, $\sqrt{n}(\hat{\gamma}-\gamma) \xrightarrow{d} N\left(0, \frac{1}{\eta(1-\eta)} \cdot \frac{e^{2\gamma}}{(e^{\gamma})^2}\right) = \cancel{N\left(0, \frac{1+\eta}{1-\eta}\right)}\ N\left(0, \frac{1}{\eta(1-\eta)}\right)$

$$g(x) = \log(x) \quad \therefore g'(x) = \frac{1}{x}$$

Lastly, it remains to check that the Hessian is -ve definite at our stationary point.

$$-n\frac{\left(\sum y_i(1-z_i)\right)^2}{(n-\sum z_i)^2}$$

~~[illegible]~~

$$H(\hat\lambda,\hat\gamma,\hat\eta) = \begin{pmatrix} \overbrace{-\frac{n}{\lambda^2}} & \left(-\sum z_i y_i\right)\left(\frac{\sum y_i(1-z_i)}{(\sum y_i\eta)(\frac{n}{\sum z_i}-1)}\right) \\ -\frac{\sum y_i(1-z_i)}{\left(\frac{n}{\sum z_i}-1\right)} & -\frac{\sum z_i}{\cancel{\sum z_i y_i}}\left(\cancel{\sum z_i y_i}\right) \end{pmatrix}$$

Retaking things from II:

evaluated at $\hat\lambda,\hat\gamma$

$$II = -n v_1^2 \frac{\left(\sum y_i(1-z_i)\right)^2}{(n-\sum z_i)^2} - 2v_1v_2\frac{\sum y_i(1-z_i)}{n-\sum z_i}\sum z_i - V_1\sum z_i$$

$$= -\sum_i\left(v_1\frac{\sum_j y_j(1-z_j)}{n-\sum_j z_j} + v_2 z_i\right)^2 < 0 \qquad \square$$

(b) By WLLN, Numerator $\xrightarrow{p} P(Y_i \geq y_0, Z_i=1)$

Denominator $\xrightarrow{p} P(Z_i=1)$

$\therefore$ $\hat\nu_1$ is consistent (by CMT)

(c) $$\sqrt{n}(\hat\nu_1-\nu) = \frac{\sqrt{n}\left(\frac{\sum \mathbb{1}\{Y_i\geq y_0, Z_i=1\}}{n} - \nu\frac{\sum \mathbb{1}\{Z_i=1\}}{n}\right)}{n^{-1}\sum \mathbb{1}\{Z_i=1\}}$$

$$= \left[\frac{1}{\sqrt{n}}\sum\psi(Y_i,Z_i)\right]\cdot\frac{\eta}{n^{-1}\sum\mathbb{1}\{Z_i=1\}}$$

$$= \left[\frac{1}{\sqrt{n}}\sum\psi(Y_i,Z_i)\right]\cdot\left(\frac{\eta-n^{-1}\sum\mathbb{1}\{Z_i=1\}}{n^{-1}\sum\mathbb{1}\{Z_i=1\}}+1\right)$$

$$= \frac{1}{\sqrt{n}}\sum\psi(Y_i,Z_i) + \underbrace{\left[\frac{1}{\sqrt{n}}\sum\psi(Y_i,Z_i)\right]}_{O_p(1)}\underbrace{\left(\frac{\eta-n^{-1}\sum\mathbb{1}\{Z_i=1\}}{n^{-1}\sum\mathbb{1}\{Z_i=1\}}\right)}_{o_p(1)}$$

$$= \underline{\frac{1}{\sqrt{n}}\sum\psi(Y_i,Z_i)} + o_p(1) \qquad \square$$

$$\xrightarrow{d} N\left(0,\frac{\nu-\nu^2}{\eta}\right)$$

$$\left(E\psi(Y_i,Z_i)^2 = \frac{1}{\eta^2}E\left[\mathbb{1}\{Y\geq y_0, Z=1\} - 2\nu\mathbb{1}\{Y\geq y_0, Z=1\} + \nu^2\eta\right]\right.$$

$$= \frac{1}{\eta^2}\left[\nu\eta - 2\nu(\nu\eta) + \nu^2\eta\right] = \frac{\nu-\nu^2}{\eta}$$

2015 Q5

It remains to show $\sqrt{n}(e^{\hat{\gamma}} - e^{\gamma}) \xrightarrow{d} N(0, e^{2\gamma} \frac{1}{\eta(1-\eta)})$

to this end, note

$$\sqrt{n}(e^{\hat{\gamma}} - e^{\gamma}) = \sqrt{n}\left(\frac{\frac{\sum y_i(1-z_i)}{n} - e^{\gamma}\left(\frac{n}{\sum z_i} - 1\right)\frac{\sum y_i z_i}{n}}{\frac{\sum y_i z_i}{n}\left(\frac{n}{\sum z_i} - 1\right)}\right)$$

$$= \sqrt{n}\left(\frac{\frac{\sum y_i(1-z_i)}{n} - e^{\gamma}\left(\frac{1}{\eta} - 1\right)\frac{\sum y_i z_i}{n}}{\frac{\sum y_i z_i}{n}\left(\frac{n}{\sum z_i} - 1\right)}\right) + \sqrt{n}\left(\frac{e^{\gamma}\frac{\sum y_i z_i}{n}\left(\frac{1}{\eta} - 1 - \left(\frac{n}{\sum z_i} - 1\right)\right)}{\frac{\sum y_i z_i}{n}\left(\frac{n}{\sum z_i} - 1\right)}\right)$$

$$\cancel{+ \sqrt{n}\left(e^{\gamma}\left(\frac{1}{\eta} - 1\right)\frac{\sum y_i z_i}{n}\left(\right.\right.}$$

$$= \frac{\sqrt{n}\left(\frac{1}{n}\sum\left[y_i(1-z_i) - e^{\gamma}\left(\frac{1-\eta}{\eta}\right)y_i z_i\right]\right)}{\left(\frac{\sum y_i z_i}{n}\right)\left(\frac{n}{\sum z_i} - 1\right)} + \frac{e^{\gamma}}{\left(\frac{n}{\sum z_i} - 1\right)}\sqrt{n}\left(\frac{1}{\eta} - \frac{n}{\sum z_i}\right)$$

$$= \frac{\sqrt{n}\left(\frac{1}{n}\sum Y_i\left(1 - z_i - e^{\gamma}\left(\frac{1-\eta}{\eta}\right)z_i\right)\right)}{\left(\frac{\sum y_i z_i}{n}\right)\left(\frac{n}{\sum z_i} - 1\right)} + \frac{\sqrt{n}\left(\frac{\sum z_i}{n} - \eta\right)}{e^{-\gamma}\eta\left(1 - \frac{\sum z_i}{n}\right)}$$

Now let $U_i = Y_i\left(1 - z_i - e^{\gamma}\left(\frac{1-\eta}{\eta}\right)z_i\right)$

so that $E U_i = E Y_i(1-z_i) - e^{\gamma}\frac{1-\eta}{\eta} E Y_i z_i = 0$ (III and IV)

$$E U_i^2 = E Y_i^2\left(1 - \left(1 + e^{\gamma}\left(\frac{1-\eta}{\eta}\right)\right)z_i\right)^2 = E\left(1 - \left(1 + e^{\gamma}\left(\frac{1-\eta}{\eta}\right)\right)z_i\right)^2 E Y_i^2 | z_i$$

$$= E\left[\left(1 - \left(1 + e^{\gamma}\left(\frac{1-\eta}{\eta}\right)\right)z_i\right)^2 \cdot \frac{2}{\lambda^2 e^{2\gamma z_i}}\right]$$

$$= \frac{2e^{2\gamma}\left(\frac{1-\eta}{\eta}\right)^2}{\lambda^2 e^{2\gamma}}\eta + \frac{2}{\lambda^2}(1-\eta) = \frac{2(1-\eta)^2}{\eta\lambda^2} + \frac{2(1-\eta)}{\lambda^2}$$

$$= \frac{(1-\eta)}{\eta\lambda^2}\left(\overset{2-2\eta+2\eta}{\cancel{\frac{2-\eta+2\eta}{2}}}\right) = \frac{2(1-\eta)\cancel{}}{\eta\lambda^2}$$

$$E\,z_i - \eta = 0 \qquad E(z_i-\eta)^2 = \text{Var}\, z_i = \eta(1-\eta)$$

and $\text{Cov}(u_i, z_i - \eta) = E\,u_i(z_i-\eta) = E\,u_i z_i$

$$= E\,Y_i(1-z_i)z_i - e^{\gamma}\left(\frac{1-\eta}{\eta}\right)E\,Y_i z_i^2$$

$$= 0 - e^{\gamma}\left(\frac{1-\eta}{\eta}\right)E\,Y_i z_i = -\frac{1-\eta}{\lambda} \quad \text{by } \text{III}$$

$\therefore$ by CLT, $\sqrt{n}\begin{pmatrix}\sum u_i/n \\ \sum z_i/n - \eta\end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\frac{2(1-\eta)}{\eta\lambda^2} & -\frac{1-\eta}{\lambda} \\ -\frac{1-\eta}{\lambda} & \eta(1-\eta)\end{pmatrix}\right)$

let ~~(z_1, z_2) follow this MVN~~ $(u_1, u_2) \sim N$ said normal

$\therefore$ by Slutsky's thm,

$$\sqrt{n}\left(e^{\hat\gamma} - e^{\gamma}\right) \xrightarrow{d} \frac{z_1\, u_1}{\frac{e^{-\gamma}(1-\eta)}{\lambda}} + \frac{u_2}{e^{-\gamma}\eta(1-\eta)}$$

this is a normal with mean 0 and variance,

$$\frac{\lambda^2}{e^{-2\gamma}(1-\eta)^2}\,2\left(\frac{(1-\eta)}{\eta\lambda^2}\right) + 2\,\frac{\lambda}{e^{-2\gamma}\eta(1-\eta)^2}\left(-\frac{1-\eta}{\lambda}\right) + \frac{1}{e^{-2\gamma}\eta^2(1-\eta)^2}\,\eta(1-\eta)$$

$$= \frac{2e^{2\gamma}\cancel{}}{\eta(1-\eta)} \cancel{+} - \frac{2e^{2\gamma}}{\eta(1-\eta)} + \frac{e^{2\gamma}}{\eta(1-\eta)}$$

$$= \frac{e^{2\gamma}}{\eta(1-\eta)} \qquad \square$$

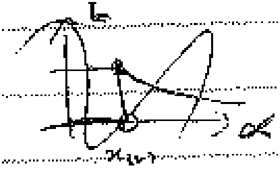
2014 Q4

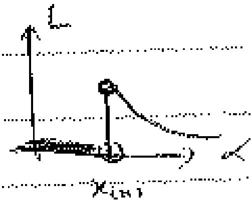
(a) $L(\alpha,\beta;\mathbf{x}) = \prod_{i=1}^{n} f(x_i|\alpha,\beta)$

$$= \beta^n \alpha^{-n\beta}\, \mathbb{1}\{0\le x_{(1)}\}\,\mathbb{1}\{x_{(n)}\le\alpha\}\left(\prod_{i=1}^{n} x_i\right)^{\beta-1}$$

firstly, note that for any value of β, as $\beta>0$, $\alpha^{-n\beta}$ is a decreasing function of α, so the likelihood ~~as a function~~ is maximised at $\alpha = X_{(n)}$. Thus $\hat{\alpha}_{MLE} = X_{(n)}$.

Irrespective of the value of β, the likelihood as a function of α looks like this:





Thus, to find the global maximum of L, it suffices to maximise $L(\hat{\alpha}_{MLE}, \beta; x)$ in β.

$$\ell(\hat{\alpha}_{MLE}, \beta; x) = n\log\beta - n\beta\log x_{(n)} + (\beta-1)\sum \log x_i$$

$$\therefore \frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - n\log x_{(n)} + \sum \log x_i$$

$$\therefore \frac{\partial^2 \ell}{\partial \beta^2} = -\frac{n}{\beta^2} < 0$$

$\therefore$ $l(\hat{\alpha}_{MLE}, \beta; x)$ is a strictly concave function of β

and so it attains its maximum at

$$\beta \quad \frac{\partial l}{\partial \beta} = 0 \implies \frac{n}{\beta} = n \log x_{(n)} - \sum \log x_i$$

$$\implies \beta = \frac{n}{n \log x_{(n)} - \sum \log x_i}$$

$$\therefore \quad \hat{\beta}_{MLE} = \frac{n}{\sum_{i=1}^{n} \log \frac{x_{(n)}}{x_i}} = \left(\frac{\sum_{i=1}^{n} \log \frac{x_{(n)}}{x_i}}{n} \right)^{-1} = \left(\log x_{(n)} - \frac{\sum \log x_i}{n} \right)^{-1}$$

CONSISTENCY:

For $\hat{\alpha}_{MLE}$, compute $P(\hat{\alpha}_{MLE} \leq x) = P(X_{(n)} \leq x)$

$$= P(X_i \leq x \ \forall i) = \prod_{i=1}^{n} \int_0^x \beta \alpha^{-\beta} \mathbb{1}_{\{0 \leq \tilde{x} \leq \alpha\}} \tilde{x}^{\beta - 1} d\tilde{x}$$

$$= \left[\beta \alpha^{-\beta} \int_0^x \tilde{x}^{\beta - 1} d\tilde{x} \right]^n = \left[\alpha^{-\beta} x^{\beta} \right]^n = \alpha^{-n\beta} x^{n\beta}$$

$$\therefore \quad P(\hat{\alpha}_{MLE} \leq x) = \begin{cases} \alpha^{-n\beta} x^{n\beta} & \text{if } x \in [0, \alpha] \\ 1 & \text{if } x > \alpha \\ 0 & \text{o/w} \end{cases} \qquad \textcircled{I}$$

$$\text{But} \quad \alpha^{-n\beta} x^{n\beta} = \left(\frac{x}{\alpha} \right)^{n\beta} \xrightarrow{n \to \infty} \begin{cases} 0 & \text{if } x = \alpha \\ 1 & \text{if } x \in [0, \alpha) \end{cases}$$

$$\therefore \quad P(\hat{\alpha}_{MLE} \leq x) \xrightarrow{n \to \infty} \mathbb{1}\{x \geq \alpha\}$$

2014 Q4

Therefore $\hat{\alpha}_{MLE} \xrightarrow{d} \alpha$ and convergence in distn to a constant implies convergence in probability to a constant

$\therefore$ $\hat{\alpha}_{MLE} \xrightarrow{p} \alpha$, as required.

As for $\hat{\beta}_{MLE}$, we compute

$\lim_{x \to 0} x^{\beta} \log x = 0$

$$E \log X_i = \int_0^{\alpha} \beta \alpha^{-\beta} x^{\beta - 1} \log x \, dx$$

$$= \cancel{\beta}\alpha^{-\beta}\left[x^{\beta} \log x\right]_0^{\alpha} - \alpha^{-\beta} \int_0^{\alpha} x^{\beta} \frac{1}{x} \, dx \quad \text{(by parts)}$$

$$= \alpha^{-\beta} \alpha^{\beta} \log \alpha - \alpha^{-\beta}\left[\frac{1}{\beta} \alpha^{\beta}\right]$$

$$= \log \alpha - \frac{1}{\beta}.$$

$\therefore$ $\dfrac{\sum \log x_i}{n} \xrightarrow{p} \log \alpha - \dfrac{1}{\beta}$ by WLLN

(we have that $E|\log X_i| = E \log X \mathbb{1}_{\{X>1\}} - E \log X \mathbb{1}_{\{X \le 1\}}$

and $-E \log X \mathbb{1}_{\{X \le 1\}} < \infty$ by the argument above,

whereas $\log X \mathbb{1}_{\{X>1\}} \in [0, \log \alpha]$ so $E \log X \mathbb{1}_{\{X>1\}} < \infty$)

Therefore $E|\log X_i| < \infty$ and WLLN applies).

But ~~$X_{(n)} \Rightarrow$~~ $X_{(n)} = \hat{\alpha}_{MLE} \xrightarrow{p} \alpha$ $\therefore$ $\log X_{(n)} \xrightarrow{p} \log \alpha$ (CMT)

Ⅱ

Thus $\log x_{(n)} - \frac{\sum \log x_i}{n} \xrightarrow{P} \frac{1}{\beta}$, and by CMT, as $\beta > 0$,

$\hat{\beta}_{MLE} = \left(\log x_{(n)} - \frac{\sum \log x_i}{n}\right)^{-1} \xrightarrow{P} \beta$, as required.

(b) Recall from I that $P(\hat{\alpha}_{MLE} \le x) = \left(\frac{x}{\alpha}\right)^{n\beta}$ for $x \in [0, \alpha]$.

$\therefore P\left(\hat{\alpha}_{MLE} \le \alpha\left(1 - \frac{x}{n\beta}\right)\right) = \left(1 - \frac{x}{n\beta}\right)^{n\beta}$ for $\alpha\left(1 - \frac{x}{n\beta}\right) \in [0, \alpha]$

~~i.e. $1 - \frac{x}{n\beta} \in [0, 1]$~~

~~i.e. $\frac{x}{n\beta} \in [0, 1]$~~

~~i.e. $x \in [0, \frac{1}{n\beta}]$~~

~~$x \in [0, n\beta]$~~

$\therefore P\left(\hat{\alpha}_{MLE} \le \alpha\left(1 - \frac{x}{n\beta}\right)\right) = \left(1 - \frac{x}{n\beta}\right)^{n\beta}$

for $\alpha\left(1 - \frac{x}{n\beta}\right) \in [0, \alpha]$ i.e. $1 - \frac{x}{n\beta} \in [0, 1]$

i.e. $\frac{x}{n\beta} \in [0, 1]$ i.e. $x \in [0, n\beta]$

$\therefore P\left(n\beta\left(1 - \frac{\hat{\alpha}_{MLE}}{\alpha}\right) \ge x\right) = \left(1 - \frac{x}{n\beta}\right)^{n\beta} \longrightarrow e^{-x}$ as $n \to \infty$

$$\therefore \boxed{n\beta\left(1 - \frac{\hat{\alpha}_{MLE}}{\alpha}\right) \xrightarrow{d} Exp(1) \text{ as } n \to \infty}$$

As for $\hat{\beta}_{MLE}$, we compute

$$n^{\gamma}(\hat{\beta}_{MLE} - \beta) = n^{\gamma}\left(\frac{1}{\log x_{(n)} - \frac{\sum \log x_i}{n}} - \beta\right)$$

$$= n^{\gamma}\left(\frac{1 - \beta \log x_{(n)} + \beta \frac{1}{n} \sum \log x_i}{\log x_{(n)} - \frac{1}{n} \sum \log x_i}\right)$$

2014 Q4

$$= n^{\gamma} \frac{1-\beta\log X_{(n)} + \beta\left[\frac{1}{n}\sum\log X_i - (\log\alpha - \frac{1}{\beta})\right] + \beta\log\alpha - 1}{\log X_{(n)} - \frac{1}{n}\sum\log X_i}$$

$$= n^{\gamma} \frac{\beta\log\frac{\alpha}{X_{(n)}} + \beta\left[\frac{1}{n}\sum\log X_i - (\log\alpha - \frac{1}{\beta})\right]}{\log X_{(n)} - \frac{1}{n}\sum\log X_i}$$

Putting $\gamma = \frac{1}{2}$, this is equal to

$$= \frac{\sqrt{n}\,\beta\log\frac{\alpha}{X_{(n)}} + \beta\sqrt{n}\left[\frac{1}{n}\sum\log X_i - (\log\alpha - \frac{1}{\beta})\right]}{\log X_{(n)} - \frac{1}{n}\sum\log X_i}$$

Now note the following

1. • By part (a), [illegible] (see II), $\log X_{(n)} - \frac{1}{n}\sum\log X_i \xrightarrow{P} \frac{1}{\beta}$

2. • $\sqrt{n}\log\frac{\alpha}{X_{(n)}} \xrightarrow{P} 0$. To see this, compute

$$P\left(\sqrt{n}\log\frac{\alpha}{X_{(n)}} \leq x\right) = P\left(\frac{\alpha}{X_{(n)}} \leq e^{x/\sqrt{n}}\right)$$

$$\cancel{X_{(n)}} = P\left(X_{(n)} \geq \alpha e^{-\frac{x}{\sqrt{n}}}\right)$$

$$= 1 - \left(\frac{\alpha e^{-\frac{x}{\sqrt{n}}}}{\alpha}\right)^{n\beta} \qquad (\text{by I})$$

$$= 1 - (e^{-x\beta})^{\sqrt{n}}$$

$$\xrightarrow{n\to\infty} 1 \quad \text{if } x > 0$$

and $\log\frac{\alpha}{X_{(n)}} \cancel{\leq}\geq 0$ a.s. as $\alpha \geq X_{(n)}$ a.s.

$\therefore \quad \sqrt{n} \log \frac{\alpha}{X_{(n)}} \xrightarrow{d} 0 \quad$ so $\quad \sqrt{n} \log \frac{\alpha}{X_{(n)}} \xrightarrow{P} 0$

(can also argue $n(X_{(n)} - \alpha) = O_p(1)$ $\therefore$ $n(\log X_{(n)} - \log \alpha) = O_p(1)$ $\therefore$ $\sqrt{n}(\log X_{(n)} - \log \alpha) = o_p(1)$)

3. By the CLT, $\sqrt{n}\left[\frac{1}{n}\Sigma \log X_i - \left(\log \alpha - \frac{1}{\beta}\right)\right] \xrightarrow{d} N(0, \mathrm{Var}\log X_1)$

(Note that $E(\log X_i)^2 = \int_0^\alpha \beta \alpha^{-\beta} x^{\beta-1} (\log x)^2 \, dx$

$$= \int_0^1 \beta \alpha^{-\beta} \alpha^{\beta-1} t^{\beta-1} (\log \alpha + \log t)^2 \alpha \, dt \qquad (\alpha t = x)$$

$$= \int_0^1 \beta t^{\beta-1} (\log \alpha + \log t)^2 \, dt$$

$$= \int_{-\infty}^0 \beta e^{s(\beta-1)} (\log \alpha + s)^2 e^s \, ds \qquad (t = e^s)$$

$$= \int_{-\infty}^0 \beta e^{s\beta} (s + \log \alpha)^2 \, ds$$

$$= (\log \alpha)^2 + 2\log\alpha \int_{-\infty}^0 \beta s e^{+\beta s} + \int_{-\infty}^0 \beta s^2 e^{+\beta s} \, ds$$

$$= (\log \alpha)^2 + \frac{2\log\alpha}{-\beta} + \frac{2}{\beta^2}$$

~~So that, by Slutsky's thm,~~ $\left((\log\alpha) - \frac{1}{\beta}\right)^2 + \frac{1}{\beta^2} > 0$

~~$\sqrt{n}(\hat{\beta}_{MLE} - \beta) \xrightarrow{d} \beta^2 N\left(0, \underbrace{(\log\alpha)^2 - \frac{2\log\alpha}{\beta} + \frac{2}{\beta^2}}\right)$~~

~~$= N\left(0, \beta^{-4}\beta^2\left((\beta\log\alpha)^2 - 2(\beta\log\alpha) + 2\right)\right)$~~

~~$= N\left(0, \beta^2\left((\beta\log\alpha - 1)^2 + 1\right)\right)$~~

2014 Q4

$$= \left(\log \alpha - \frac{1}{\beta}\right)^2 + \frac{1}{\beta^2}$$

$$\therefore \quad \operatorname{Var} \log X_1 = \left(\log \alpha - \frac{1}{\beta}\right)^2 + \frac{1}{\beta^2} - \left(\log \alpha - \frac{1}{\beta}\right)^2$$

$$= \frac{1}{\beta^2}$$

Thus, putting the pieces together, by Slutsky's,

$$\sqrt{n}\left(\hat{\beta}_{MLE} - \beta\right) \xrightarrow{d} \beta^2 N\left(0, \frac{1}{\beta^2}\right) = N(0, \beta^2)$$

Note also that $\hat{\alpha} \perp\!\!\!\perp \beta$ $\hat{\alpha} \perp\!\!\!\perp \hat{\beta}$ by Basu's thm ($\hat{\alpha}$ is CS for α for any fixed β, $\hat{\beta}$ ancillary)

(c) Thus $$\frac{\sqrt{n}\left(\hat{\beta}_{MLE} - \beta\right)}{\beta} \xrightarrow{d} N(0,1)$$

$$\therefore \quad P\left(-z_{1-\frac{\alpha}{2}} \leq \frac{\sqrt{n}\left(\hat{\beta}_{MLE} - \beta\right)}{\beta} \leq z_{1-\frac{\alpha}{2}}\right) \xrightarrow{n \to \infty} 1 - \alpha \quad (\alpha = 0.05)$$

~~$\therefore \quad P\left(\sqrt{n}\,(1 - z_{1-\frac{\alpha}{2}} \leq \sqrt{n}\,\frac{\hat{\beta}_{MLE}}{\beta} \leq z_{1-\frac{\alpha}{2}} + 1\right) \longrightarrow 1 - \alpha$~~

$$\therefore \quad P\left(\sqrt{n}\,\frac{\hat{\beta}_{MLE}}{(1 + z_{1-\frac{\alpha}{2}})} \leq \beta \leq \sqrt{n}\,\frac{\hat{\beta}_{MLE}}{1 - z_{1-\frac{\alpha}{2}}}\right) \xrightarrow{n \to \infty} 1 - \alpha$$

~~∴ 95% asymptotic C.I is~~

~~$\left(\sqrt{n}\,\frac{\hat{\beta}_{MLE}}{1+z_{1-\frac{\alpha}{2}}}, \sqrt{n}\,\frac{\hat{\beta}_{MLE}}{1-z_{1-\frac{\alpha}{2}}}\right)$~~

~~$\therefore P\left(-\sqrt{n}-z_{1-\frac{\alpha}{2}} \le \sqrt{n}\,\frac{\hat{\beta}_{MLE}}{\beta} \le \sqrt{n}\right.$~~

$$\therefore P\left(+1-z_{1-\frac{\alpha}{2}}/\sqrt{n} \le \frac{\hat{\beta}_{MLE}}{\beta} \le \frac{z_{1-\frac{\alpha}{2}}}{\sqrt{n}}+1\right) \longrightarrow 1-\alpha$$

$$\therefore P\left(\frac{\hat{\beta}_{MLE}}{1+z_{1-\frac{\alpha}{2}}/\sqrt{n}} \le \beta \le \frac{\hat{\beta}_{MLE}}{1\mp z_{1-\frac{\alpha}{2}}/\sqrt{n}}\right) \longrightarrow 1-\alpha$$

∴ 95% C.I is

$$\left(\frac{\hat{\beta}_{MLE}}{1+z_{1-\frac{\alpha}{2}}/\sqrt{n}}, \frac{\hat{\beta}_{MLE}}{1-z_{1-\frac{\alpha}{2}}/\sqrt{n}}\right)$$

and we have already shown its Probability $\longrightarrow 1-\alpha = 0.95$.

Alternatively, by Slutsky's $\frac{\sqrt{n}(\hat{\beta}_{MLE}-\beta)}{\hat{\beta}_{MLE}} \xrightarrow{d} N(0,1)$

$$\therefore P\left(-z_{1-\frac{\alpha}{2}} \le \frac{\sqrt{n}(\hat{\beta}_{MLE}-\beta)}{\hat{\beta}_{MLE}} \le z_{1-\frac{\alpha}{2}}\right) \longrightarrow N(0,1)$$

$\therefore \left(\hat{\beta}_{MLE}\left(1-\frac{z_{1-\frac{\alpha}{2}}}{\sqrt{n}}\right), \hat{\beta}_{MLE}\left(1+\frac{z_{1-\frac{\alpha}{2}}}{\sqrt{n}}\right)\right)$ is a $1-\alpha$ C.I.

2014 Q2

(a) $p_\lambda(\tilde{x}) = \lambda^n e^{-\lambda \Sigma x_i}$

$= \exp\{-\lambda \Sigma x_i + n \log \lambda\}$

this is an exponential family w. natural parameter $\lambda \in (0,\infty)$.
As $(0,\infty)$ has non-empty interior, Σx_i is CS and MS.

(b) $\phi = P_\lambda(X_1 > x) = \int_x^\infty \lambda e^{-\lambda \tilde{x}} d\tilde{x} = \left[-e^{-\lambda \tilde{x}}\right]_x^\infty = e^{-\lambda x}$

So $\phi = e^{-\lambda x}$.

We know $I(\lambda) = n\lambda^{-1}$ (class results)

Also, by class results, $I(g(\lambda)) = \frac{I(\lambda)}{g'(\lambda)^2}$, so letting $g(\lambda) = e^{-\lambda x}$

$\lambda = -\frac{\log \phi}{x}$

$\therefore \quad I(\phi) = \frac{n\lambda^{-1}}{x^2 e^{-2\lambda x}}$ ~~letting g(λ) =~~

$= \frac{-n}{x(\log\phi)\phi^2}$

$\boxed{= \frac{-n}{x\phi^2 \log\phi}}$ $\phi \in (0,1)$

(c) Lehman-Scheffe Thm: Suppose $\delta(X)$ is an unbiased estimator of $g(\theta)$ which is a function of ~~the~~ C.S. ~~statistic~~ T.
Then $\delta(X)$ is UMVUE.

∴ Only need to check that $\hat{\phi}_n$ is unbiased

(from a) T_n is CS for λ ∴ T_n is CS for ϕ,
as ϕ is a 1-1 reparametrisation).

$$E\,\tilde{\phi}_n = E\left(1-\frac{x}{T_n}\right)^{n-1} \mathbb{1}_{\{T_n \geq x\}}$$

$$= \int_0^\infty \left(1-\frac{x}{t}\right)^{n-1} \mathbb{1}_{\{t \geq x\}}\, f(t)\, dt$$

But $T_n = \sum_i^n x_i \sim \text{Gamma}(n, \lambda)$

$$= \int_x^\infty \left(1-\frac{x}{t}\right)^{n-1} \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}\, dt$$

$$= \int_x^\infty (t-x)^{n-1} e^{-\lambda t} \frac{\lambda^n}{\Gamma(n)}\, dt \,. \qquad \text{let } s = t-x$$

~~$= \int_0^\infty s^{n-1} e^{-\lambda s}\, ds\, e^{-\lambda x}$~~

$$= \int_0^\infty s^{n-1} e^{-\lambda s - \lambda x} \frac{\lambda^n}{\Gamma(n)}\, ds$$

$$= e^{-\lambda x} \quad \text{as required}$$

Referring back to the proof of CRLB, if CRLB were attained,
then $p_\phi(\vec{x})$ would be a 1-parameter exp. fam. with natural
sufficient statistic $\tilde{\phi}_n(\vec{x})$. ~~However, the~~ This is not the case here.
(natural sufficient statistic is $T_n(x)$, which is not a function of $\tilde{\phi}_n$).

(d) $\tilde{\phi}_n$ is unbiased. T_n is C.S. ∴ $E(\tilde{\phi}_n | T_n)$ is UMVUE. By uniqueness: $E\,\hat{\phi} | T_n = \tilde{\phi}$

2014 Q3

(a) The Bayes estimate θ^* will minimise the Bayes risk: $r(\pi, \hat{\theta}) = E_{(\theta, X)} \, \ell(\theta, \hat{\theta}(X))$

$$= E\left[E\left[\frac{(\theta - \hat{\theta})^2}{\theta} \middle| X\right]\right]$$

~~$= E[\theta$~~

$$= E\left[E[\theta | X] - 2\hat{\theta}(X) + \hat{\theta}(X)^2 E[\tfrac{1}{\theta} | X] \right]$$

To minimise this expectation in $\hat{\theta}$, it suffices to minimise the inner expectation pointwise in $\hat{\theta}$. But this is a quadratic, so we should choose $\theta^*(X) = \frac{1}{E[\frac{1}{\theta}|X]}$ □.

The Bayes risk for this estimator is

$$b^* = r(\pi, \theta^*) = E\left[E[\theta|X] - 2 \frac{1}{E[\frac{1}{\theta}|X]} + \frac{1}{E[\frac{1}{\theta}|X]} \right]$$

$$= E\theta - E\left[\frac{1}{E[\frac{1}{\theta}|X]}\right]$$

(b) In this case, the Bayes estimator is

$$d^*(X) = \frac{E\left[\frac{1}{1-\theta}\,|X\right]}{E\left[\frac{1}{\theta(1-\theta)}\,|X\right]}$$ by the same reasoning as before.

Now compute

$$L(\theta;X) = \binom{n}{X}\theta^X(1-\theta)^{n-X} \qquad \pi(\theta) = \mathbb{1}_{[0\le\theta\le1]}$$

$$\therefore\ \pi(\theta|X) \propto \theta^X(1-\theta)^{n-X}$$

$$\therefore\ \theta|X \sim \text{Beta}(X+1,\, n-X+1), \quad \pi(\theta|X) = B(X+1,n-X+1)\,\theta^X(1-\theta)^{n-X}.$$

$$\therefore\ E\left[\tfrac{1}{1-\theta}\,|X\right] = \int_0^1 B(X+1,n-X+1)^{-1}\theta^X(1-\theta)^{n-X-1}\,d\theta = \frac{B(X+1,n-X+1)^{-1}}{B(X+1,n-X)^{-1}}$$

$$= \left(\frac{X!(n-X)!/(n+1)!}{X!(n-X-1)!/n!}\right)^{-1} = \left(\frac{n-X}{n+1}\right)^{-1}$$

$$E\left[\frac{1}{\theta(1-\theta)}\,|X\right] = \int_0^1 B(X+1,n-X+1)^{-1}\theta^{X-1}(1-\theta)^{n-X-1}\,d\theta = \left(\frac{B(X+1,n-X+1)}{B(X,n-X)}\right)^{-1}$$

$$= \left(\frac{X!(n-X)!/(n+1)!}{(X-1)!(n-X)!/(n-1)!}\right)^{-1} = \left(\frac{X(n-X)}{(n+1)n}\right)^{-1}$$

$$\therefore\ d^*(X) = \left(\frac{\frac{n-X}{n+1}}{\frac{X(n-X)}{n(n+1)}}\right)^{-1} = \left(\frac{n}{X}\right)^{-1} = \frac{X}{n}.$$ ⟶ (careful with cases X=0 or n!)

$$\therefore\ R(\theta,d^*) = E_\theta\frac{(\theta-d)^2}{\theta(1-\theta)} = \frac{1}{\theta(1-\theta)}E_\theta\left(\theta-\frac{X}{n}\right)^2 = \frac{1}{n^2\theta(1-\theta)}\overbrace{E_\theta(X-n\theta)^2}^{\text{Var}(\text{Bin}(n,\theta)) = n\theta(1-\theta)} = \boxed{\frac{1}{n}}$$

Thus d^* is a Bayes estimator with constant risk, so it is minimax

By class results: the marginal of X puts mass on all $\{0,\dots,n\}$ ∴ marginal dominates the conditional ∴ d^* is unique Bayes ∴ d^* is admissible.

2011 Q5

(a) $L(\theta;X) = \prod_{i=1}^{n} (2\pi)^{-\frac{k}{2}} \exp\{-\frac{1}{2}\|X_i - \theta\|^2\}$

$\therefore \ell(\theta;X) = \text{constant} - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{k}(X_{ij}-\theta_j)^2$

$= \text{constant} - \frac{1}{2}\sum_{j=1}^{k}\sum_{i=1}^{n}(X_{ij} - \bar{X}_{\cdot j} + \bar{X}_{\cdot j} - \theta_j)^2$

$= \text{constant} - \frac{n}{2}\sum_{j=1}^{n}(\bar{X}_{\cdot j} - \theta_j)^2$

which splits into the k components of θ.

Clearly each quadratic components is maximized at $\theta_j = \bar{X}_{\cdot j}$.

But as $\theta \in \{v \in \mathbb{R}^k : \sup_i |v_i| = 1\}$, we know that each component satisfies $|\theta_j| \leq 1$, and at least one component satisfies $|\theta_j| = 1$. Therefore we split into cases.

~~Case 1: At least one $\exists$~~

~~$\exists j^*$ s.t. $|\bar{X}_{\cdot j^*}| \geq 1$~~

~~If $X_{\cdot j^*}$~~

Case 1: $\exists \ell \in \{1,\dots,k\}$ s.t. $|\bar{X}_{\cdot \ell}| \geq 1$.

If $\bar{X}_{\cdot \ell} \leq -1$, then $\ell(\theta;X)$ is decreasing in $\theta_\ell \in [-1,1]$ (regardless of the values of θ_i, $i \neq \ell$)

∴ $l(\theta;X)$ is maximal at $\hat{\theta}_\ell = -1$.

If $\bar{X}_{\cdot\ell} \geq 1$, similarly $l(\theta;X)$ is increasing ~~for~~ in $\theta_\ell \in [-1,1]$ ∴ $l(\theta;X)$ is maximal at $\hat{\theta}_\ell = 1$.

Similarly, for each other component θ_j, $j \neq \ell$,

$$\hat{\theta}_j = \begin{cases} -1 & \text{if } \bar{X}_{\cdot j} \leq -1 \\ \bar{X}_{\cdot j} & \text{if } \bar{X}_{\cdot j} \in [-1,1] \\ 1 & \text{if } X_{\cdot j} \geq 1 \end{cases}$$

∴ maximizing $l(\theta,X)$ over $\theta \in \{v \in \mathbb{R}^k : \sup_i |v_i| \leq 1\}$ yields a maximizer ~~$\hat{\theta}_j$ that is~~ $\hat{\theta}$ that satisfies $\sup_i |\hat{\theta}_i| = 1$, so this is the desired MLE.

Case 2: $|\bar{X}_{\cdot j}| < 1 \quad \forall j$

Again, maximizing $l(\theta;X)$ over $\theta \in \{v \in \mathbb{R}^k . \sup_i |v_i| \leq 1\}$ can be achieved by setting all the quadratics to 0 at $\hat{\theta}_j = \bar{X}_{\cdot j}$ $\forall j$, which is in fact the global maximum of $l(\theta;X)$ in $\mathbb{R}^k$. To find the maximum in our desired region, the unit l_∞ sphere,

2014 Q5

we must set at least one $\hat{\theta}_j$ to ± 1.

By doing so, we will increase $l(\theta; x)$ by the squared distance from $\bar{X}_{\cdot j}$ to ± 1. Therefore, the MLE must set $\hat{\theta}_j = 1 \cdot \text{sign}(\bar{X}_{\cdot j})$ for $j = \underset{i}{\arg\min}\ 1 - |\bar{X}_{\cdot i}|$.

~~$\hat{\theta}_{MLE}$~~

$$\therefore \quad \hat{\theta}_j = \begin{cases} \bar{X}_{\cdot j} & \forall \ j \neq \underset{i}{\arg\min}\ 1 - |\bar{X}_{\cdot i}| \\ 1 \cdot \text{sign}(\bar{X}_{\cdot j}) & \text{if } j = \underset{i}{\arg\min}\ 1 - |\bar{X}_{\cdot i}| \end{cases}$$

Now suppose $\theta = (1, \frac{1}{2}, \dots, \frac{1}{2})$.

Then $\bar{X}_{\cdot 1} \xrightarrow{p} 1$ and $\bar{X}_{\cdot j} \xrightarrow{p} \frac{1}{2}$ $\forall j \geq 2$.

$\therefore$ w.h.p. $|\bar{X}_{\cdot j}| < 1$ $\forall j \geq 2$ and $1 = \underset{j}{\arg\min}\ 1 - |\bar{X}_{\cdot j}|$

$\therefore$ w.h.p. $\hat{\theta}_j = \bar{X}_{\cdot j}$ $\forall j \geq 2$ and $\underbrace{\hat{\theta}_1 = 1}$

either $\bar{X}_1 \geq 1$ so then $\hat{\theta}_1 = 1$ by case 1, or $\bar{X}_1 < 1$ so $1 = \underset{j}{\arg\min}\ 1 - |\bar{X}_{\cdot j}|$ w.h.p. and so $\hat{\theta}_1 = 1$ by case 2.

~~and $\hat{\theta}_1 = \bar{X}_{\cdot 1} - (\bar{X}_{\cdot 1} - 1)_+$~~

but θ $\bar{X}_{\cdot j} \xrightarrow{p} \frac{1}{2}$ $\forall j \geq 2$ ~~and~~

~~$\bar{X}_{\cdot 1} - (\bar{X}_{\cdot 1} - 1)_+ \xrightarrow{p} 1 - (1-1)_+ = 1$ (CMT)~~

$\therefore$ $\hat{\theta}_{MLE}$ is consistent $\square$

(b) By CLT, $\sqrt{n}\left(\begin{pmatrix}\bar X_{\cdot 1}\\ \bar X_{\cdot 2}\\ \vdots \\ \bar X_{\cdot k}\end{pmatrix} - \theta\right) \xrightarrow{d} N(0, I_k)$

~~But~~ Also, ~~$*\ \hat\theta_1 = \bar X_{\cdot 1} - (\bar X_{\cdot 1} - 1)_+$~~ w.h.p. $\hat\theta_1 = 1$ and

~~$\sqrt{n}(\hat\theta_1 - 1) = \sqrt{n}(\bar X_{\cdot 1} - 1) - \sqrt{n}(\bar X_{\cdot 1} - 1)_+ \xrightarrow{d} Z - Z_+$ where $Z \sim N(0,1)$~~

~~[illegible]~~

~~or equivalently $\sqrt{n}(\hat\theta_1 - 1) \xrightarrow{d} \begin{cases} 0 & \text{w.p. } 1/2 \\ N(0,1) \text{ conditional on } N(0,1)<0 & \text{w.p. } 1/2 \end{cases}$~~

and w.h.p.

~~and~~ $\sqrt{n}(\hat\theta_j - 1/2) = \sqrt{n}(\bar X_{\cdot j} - 1/2)$ ~~$N(0,1)$~~ $\forall j \geq 2$.

and $\sqrt{n}(\bar X_{\cdot j} - \frac{1}{2}) \xrightarrow{d} N(0,1)$.

(as $\hat\theta_i \perp\!\!\!\perp \hat\theta_j$ $\forall i \neq j$ ~~(as they depend only on the independent~~

(as they are functions of the independent components $\bar X_{\cdot i}$ and $\bar X_{\cdot j}$)),

~~It follows that $\sqrt{n}(\hat\theta - \theta) \xrightarrow{d} N\left(\vec{0}, \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & & I_{k-1}\end{pmatrix}\right)$~~

But $\bar X_{\cdot i} \perp\!\!\!\perp \bar X_{\cdot j}$ $\forall i \neq j$.

~~$\sqrt{n}(\hat\theta - \theta) \xrightarrow{d} \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_k \end{pmatrix} - \begin{pmatrix} (Z_1)_+ \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ where $Z_1, \dots, Z_k \overset{iid}{\sim} N(0,1)$~~

∴ W.h.p. $\sqrt{n}(\hat\theta - \theta) = \sqrt{n}\begin{pmatrix} 0 \\ \bar X_{\cdot 2} - \frac{1}{2} \\ \vdots \\ \bar X_{\cdot k} - \frac{1}{2}\end{pmatrix}$ and $\sqrt{n}\begin{pmatrix} 0 \\ \bar X_{\cdot 2} - 1/2 \\ \vdots \\ \bar X_{\cdot k} - 1/2 \end{pmatrix} \xrightarrow{d} N\left(0, I\begin{pmatrix} 0 & \cdots & 0 \\ 0 & I_{k-1}\end{pmatrix}\right)$

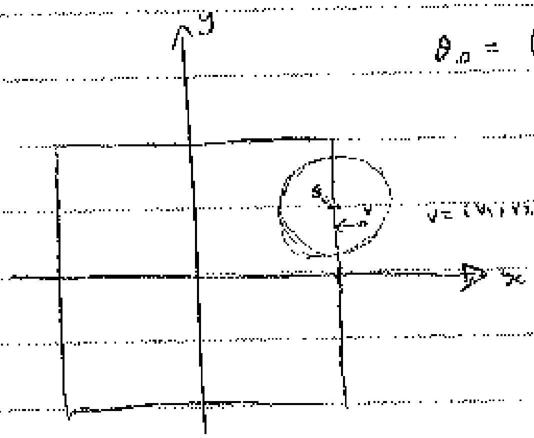
Therefore $\sqrt{n}(\hat\theta - \theta) \xrightarrow{d} N\left(\vec{0}, \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & & I_{k-1}\end{pmatrix}\right)$

2014 Q5

$$\hat{\theta}_{MLE} = \underset{\theta \in C}{\operatorname{argmin}} \|\bar{X} - \theta\|_2^2$$

$C = \{ v \in \mathbb{R}^k : \|v\|_\infty = 1 \}$

$\theta_0 = (1, \frac{1}{2}, \ldots, \frac{1}{2})$



$v = (v_1, v_2) \Rightarrow P_C(v) = (1, v_2)$ locally

$\therefore \nabla P_C(v) = \begin{pmatrix} 0 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ locally

Let $P_C(x)$ be the map that projects $\vec{x} \in \mathbb{R}^k$ to C.

~~By CMT, $\hat{\theta}_{MLE} \overset{w.h.p.}{=} P_C(\bar{X})$~~

By WLLN, $\bar{X} \xrightarrow{P} \vec{\theta}_0$

$\therefore$ w.h.p. $\hat{\theta}_{MLE} = P_C(v)$

But MLE is $\hat{\theta}_{MLE} = P_C(v)$

$\therefore$ w.h.p. $\hat{\theta}_{MLE} = (1, \bar{X}_2, \ldots, \bar{X}_k)^T$

$\therefore$ By CMT, $\hat{\theta}_{MLE} \xrightarrow{P} \theta_0$

By Δ-method $\sqrt{n}(\hat{\theta}_{MLE} - \theta_0) \xrightarrow{d} N\left(\mathbf{0}, \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & I_{k-1} & \\ 0 & & \end{pmatrix}\right)$

2013 Q1

Correct likelihood is:

$$p_\theta(\vec{x}, z) = \frac{1}{2}\theta^{r}(1-\theta)^{s}\left[\mathbb{1}_{\{r+s=n, z=1\}} + \mathbb{1}_{\{r=k, z=2\}}\right]$$

Consider the likelihood ratio:

$$\frac{p_\theta(x_1, z_1)}{p_\theta(x_2, z_2)} = \theta^{r_1-r_2}(1-\theta)^{s_1-s_2}\left[\frac{\mathbb{1}_{\{r_1+s_1=n, z_1=1\}} + \mathbb{1}_{\{r_1=k, z_1=2\}}}{\mathbb{1}_{\{r_2+s_2=n, z_2=1\}} + \mathbb{1}_{\{r_2=k, z_2=2\}}}\right]$$

This is independent of θ iff $(r_1, s_1) = (r_2, s_2)$

$\therefore$ (R, S) is MS.

Therefore:

(a) R is not suff ~~(not a function of~~

((R,S) is not a func of R, e.g. $(R, S, Z) = (r, n-r, 1)$ or $(r, n-r, 2)$)

(b) (Z, R) is ~~MS~~ sufficient, but NOT M.S.

$(R, S) = (R, n-R)\mathbb{1}_{\{Z=1\}} + (R, k)\mathbb{1}_{\{Z=2\}}$

~~$(Z,R) =$~~ so (R,S) is a func of (Z,R).

However, $(R,S) = (r, n-r)$ can arise from

$(Z,R) = (1, r)$ or $(Z,R) = (2, r)$

so ~~(R,S~~ (Z,R) is NOT a func of (R,S)

(c) (R,S) is the M.S. statistic

(d) $(Z, R+S)$ is NOT suff because:

(R, S) is not a func of $(Z, R+S)$

e.g. ~~$(Z, R+S) =$~~ $(R, S) = (1, n-1)$ or $(2, n-2)$

can both correspond to $(Z, R+S) = (1, n)$.

(e) (Z, R, S) is sufficient but NOT M.S.

Clearly (R, S) is a func of (Z, R, S)

however, (Z, R, S) is NOT a func of (R, S):

$(Z, R, S) = (1, r, n-r)$ or $(2, r, n-r)$ both lead to $(R, S) = (r, n-r)$

(f) $Y = 2R + Z$ if $R \neq n-k$, $Y = 0$ if $R = n-k$.

This is M.S.

$$(R, S) = (n-k, k)\mathbb{1}_{\{Y=0\}} + \mathbb{1}_{\{Y \text{ odd}\}}\left(\frac{Y-1}{2}, n - \frac{Y-1}{2}\right) + \mathbb{1}_{\{Y \text{ even}\}}\left(\frac{Y-2}{2}, k\right)$$

& $Y = 0\,\mathbb{1}_{\{R=n-k\}} + \mathbb{1}_{\{R \neq n-k\}}\left(2R + \mathbb{1}_{\{S=n-R\}} + 2\,\mathbb{1}_{\{S=k\}}\right)$

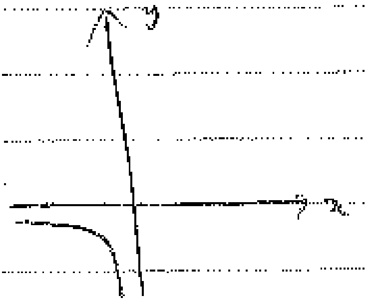
2013 Q1

$$L(x,z) = \left[\theta^{k}(1-\theta)^{n-k}\right]^{2-z}\left[\theta^{k}(1-\theta)^{k}\right]^{z-1}$$

$$\therefore \; L(x,z) \propto \exp\left\{k\log\theta + (k-(n-k))(z-2)\log(1-\theta)\right\}$$

~~note~~ $\eta(\theta) = (\log\theta, \log(1-\theta))$

if $x = \log\theta$, then $\log(1-\theta) = \log(1-e^{x})$



$\therefore \; \exists \; v_0, v_1, v_2 \in \bar{\eta}$

and $v_1 - v_0$, $v_2 - v_0$ are linearly independent

2013 Q7

(a) $L(A,B;Y) = (2\pi v)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2v}\Sigma(Y_i - A - Bx_i)^2\right\}$

$L(0,0;Y) = (2\pi v)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2v}\Sigma Y_i^2\right\}$

to maximise L, pick

$$\begin{pmatrix}\hat{A}\\ \hat{B}\end{pmatrix} = (X^TX)^{-1}X^TY \quad \text{where} \quad X = \begin{pmatrix}1 & x_1\\ 1 & x_2\\ \vdots & \vdots\\ 1 & x_n\end{pmatrix}$$

$\therefore \hat{A} = \frac{\Sigma Y_i}{n} \qquad \hat{B} = \frac{\Sigma x_i Y_i}{\Sigma x_i^2} = \frac{\Sigma x_i Y_i}{n}$

$\therefore \sup_{H_1} L(A,B;Y) = (2\pi v)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2v}\Sigma\left(Y_i - \frac{\Sigma Y_j}{n} - x_i\frac{\Sigma x_j Y_j}{n}\right)^2\right\}$

$\therefore W = -2\log\frac{L_0}{\hat{L}} = 2\log\hat{L} - 2\log L_0$

$= 2\frac{1}{2v}\Sigma Y_i^2 - 2\frac{1}{2v}\Sigma\left(Y_i - \frac{\Sigma Y_j}{n} - x_i\frac{\Sigma x_j Y_j}{n}\right)^2$

$= 2\cdot\frac{1}{\cancel{2}v}\cdot\Sigma Y_i\left(\frac{\Sigma Y_j}{n} + x_i\frac{\Sigma x_j Y_j}{n}\right) - \frac{1}{\cancel{2}v}\Sigma\left(\frac{\Sigma Y_j}{n} + x_i\frac{\Sigma x_j Y_j}{n}\right)^2$

$= 2\frac{1}{v}\; n\bar{Y}^2 + 2n\left(\frac{\Sigma x_j Y_j}{n}\right)^2 - \frac{1}{\cancel{2}v}\Sigma\; \bar{Y}^2 + 2x_i\bar{Y}\frac{\Sigma x_j Y_j}{n} + x_i^2\left(\frac{\Sigma x_j Y_j}{n}\right)^2$

$= 2\frac{1}{v}n\bar{Y}^2 + 2n\left(\frac{\Sigma x_j Y_j}{n}\right)^2 - \frac{n}{\cancel{2}v}\cancel{2}\bar{Y}^2 - \frac{n}{v}\left(\frac{\Sigma x_j Y_j}{n}\right)^2$

$= \frac{n}{v}\bar{Y}^2 + \frac{n}{v}\left(\frac{\Sigma x_j Y_j}{n}\right)^2 = n\frac{\bar{Y}}{\sqrt{v}}$

now note that $\frac{\bar{Y}}{\sqrt{v}}$ $\bar{Y}$ under H_0,

$\bar{Y} \sim N(0, \frac{V}{n})$ $\therefore$ $\frac{\sqrt{n}}{\sqrt{V}} \bar{Y} \sim N(0,1)$ $\therefore$ $\frac{n}{V} \bar{Y}^2 \sim \chi^2_1$

and $\Sigma x_j Y_j \sim N(0; V \Sigma x_j^2) = N(0, nV)$

$\therefore$ $\frac{\Sigma x_j Y_j}{\sqrt{nV}} \sim N(0,1)$ $\therefore$ $\frac{(\Sigma x_j Y_j)^2}{nV} \sim \chi^2_1$

Lastly, $\operatorname{Cov}(\bar{Y}, \Sigma x_j Y_j) =$

$= \frac{1}{n} \operatorname{Cov}(\Sigma Y_j, \Sigma x_j Y_j) = \frac{1}{n} \sum_{i,j} x_j \operatorname{Cov}(Y_i, Y_j)$

$= \frac{1}{n} \Sigma x_j V = 0$.

$\therefore$ $\bar{Y} \perp\!\!\!\perp \Sigma x_j Y_j$ (as they are uncorrelated MVN)

$\therefore$ $W \sim \chi^2_2$ under H_0.

(b) In this case, $L(0,0,1) \propto (2\pi)^{-\frac{n}{2}} \exp\{-\frac{1}{2} \Sigma Y_i^2\}$,

whereas $\sup_{H_1} L(A,B,V;Y) = (2\pi\hat{V})^{-\frac{n}{2}} \exp\{-\frac{1}{2\hat{V}} \Sigma (Y_i - \hat{A} - x_i \hat{B})^2\}$

$\hat{A} = \bar{Y}$ and $\hat{B} = \overline{xY}$ by same argument,

$\therefore$ $\hat{V} = \frac{RSS}{n} = \frac{\Sigma(Y_i - \bar{Y} - x_i(\overline{xY}))^2}{n} =$

$\therefore$ $\sup_{H_1} L(A,B,V) = \frac{n}{2} \quad 2\pi \quad \Sigma$

$\therefore$ $W = \Sigma Y_i^2 - n - n \log \frac{\Sigma(Y_i - \bar{Y} - x_i(\overline{xY}))^2}{n}$

2013 Q5

(a) $L(\mu,\sigma^2; X) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\Sigma(X_i-\mu)^2\right\}$ $\quad \mu \in \mathbb{Z}$, $\sigma^2 > 0$.

$\therefore \; l(\mu,\sigma^2; X) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\Sigma(X_i-\bar{X})^2 - \frac{n}{2\sigma^2}(\bar{X}-\mu)^2$

Let $g(x)$ denote the closest integer to x for any $x \in \mathbb{R}$,

and $\forall \; x \in \mathbb{Z} + \frac{1}{2}$, define $g(x) = x + \frac{1}{2}$. (somewhat arbitrarily)

Irrespective of the value of σ^2, $l(\mu,\sigma^2, X)$ is a

quadratic in μ which is symmetric about $\mu = \bar{X}$

and ~~tends as $\mu \to \pm\infty$~~ is strictly concave

$\therefore \; \hat{\mu} = g(\bar{X})$. $\quad \square$

(b) $P(\hat{\mu} \neq \mu) = P(g(\bar{X}) \neq \mu) =$

$$= P\left(\bar{X} \notin \left(\mu - \tfrac{1}{2}, \mu + \tfrac{1}{2}\right)\right)$$

$$= 1 - P\left(\mu - \tfrac{1}{2} < \bar{X} < \mu + \tfrac{1}{2}\right)$$

$$= 1 - P\left(-\frac{\sqrt{n}}{2} < \sqrt{n}(\bar{X}-\mu) < \frac{\sqrt{n}}{2}\right)$$

$$= 1 - \left(\Phi\left(\frac{\sqrt{n}}{2}\sigma\right) - \Phi\left(-\frac{\sqrt{n}}{2}\sigma\right)\right)$$

$$= 2 - 2\Phi\left(\frac{\sigma\sqrt{n}}{2}\right) = 2\Phi\left(-\frac{\sigma\sqrt{n}}{2}\right) \longrightarrow 0 \text{ as } n \to \infty$$

hence $\hat{\mu}$ is consistent for μ.

Moreover note that, for any $\alpha > 0$, $t > 0$,

$$P(|n^\alpha(\hat{\mu}_n-\mu)| > t) = P\left(|\hat{\mu}_n-\mu| > \frac{t}{n^\alpha}\right)$$

$$\leq P(\hat{\mu}_n \neq \mu) \xrightarrow{n\to\infty} 0 \quad \text{by the previous result}$$

the same holds replacing n^α by any $a(n) \uparrow \infty$

i.e. $a(n)(\hat{\mu}_n-\mu) \xrightarrow{P} 0$ for any $\{a(n)\}$

$\nearrow$ See 6.33 TPE

2011 Q2

Suppose F and G have densities, $f(x) = \frac{dF}{dx}$ & $g(x) = \frac{dG}{dx}$.

Then $\theta = \int G(x)\, dF(x) = \int G(x) f(x)\, dx = E_{X \sim f(x)} \left[G(X) \right]$

(law of the unconscious statistician)

Let $X_1, \dots, X_n \sim F$, $Y_1, \dots, Y_m \sim G$

We know $(X_{(1)}, \dots, X_{(n)}, Y_{(1)}, \dots, Y_{(m)})$ is C.S. for (F, G)

Proof: By class results, $(X_{(1)}, \dots, X_{(n)})$ is CS for $\{F\}$
and $(Y_{(1)}, \dots, Y_{(m)})$ is CS for $\{G\}$.

Thus,

Now, if we knew $G(y)$, then $\frac{1}{n} \sum_{i=1}^{n} G(X_i)$ would be an unbiased estimate for θ $E_{X \sim f(x)} G(X)$.

But we don't, so estimate $\hat{G}(y) = \frac{1}{m} \sum_{j=1}^{m} \mathbb{1}\{Y_j \le y\}$.

Putting the pieces together,

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} \left[\frac{1}{m} \sum_{j=1}^{m} \mathbb{1}\{Y_j \le X_i\} \right]$$

is an unbiased estimator of $\hat{\theta}$ which is a function of the C.S. statistic. $\therefore$ $\hat{\theta}$ is UMVUE.

$\left(E \hat{\theta} = E\, \mathbb{1}\{Y_j \le X_i\} = P(Y_1 \le X_1) = \int P(Y_1 \le X_1 \mid X_1 = x)\, dF(x) = \int G(x)\, dF(x) \right)$

remains to show that ~~$(X_{(1)}, \dots, Y_{(m)})$~~

$(X_{(1)}, \dots, X_{(n)}, Y_{(1)}, \dots, Y_{(m)})$ is c.s. for

$\{F, G :$ F and G are p. measures on $\mathbb{R}\}$.

By class results ? $\{X_{(1)}, \dots, X_{(n)}\}$ is c.s. for $\{F\}$.

$E\, h(X_{(1)}, \dots, X_{(n)}, Y_{(1)}, \dots, Y_{(m)}) = 0 \quad \forall F, G$

$\Rightarrow \underset{X \sim F}{E} \; \underset{Y \sim G}{E} \, h(X_{(1)}, \dots, X_{(n)}, Y_{(1)}, \dots, Y_{(m)}) = 0 \quad \forall F, G$

$\Rightarrow E_{Y \sim G} h(X_{(1)}, \dots, X_{(n)}, Y_{(1)}, \dots, Y_{(m)}) = 0$ a.s. F $\forall G, F$

$\Rightarrow h = 0$ a.s. F, G $\forall G, F$.

$\int \mathbb{1}_A f(x_{(1)}, \dots, x_{(n)})\, dx = 0$

... Dynkin's lemma

In general, if if $\{P_1$ T_1 is c.s. for $\{P_1\}$

T_2 is c.s. for $\{P_2\}$

then $\{T_1, T_2\}$ is CS for $\{P_1, P_2\}$

2011 Q3

$$L(\mu_1, \mu_2, \sigma^2 ; X) = (2\pi\sigma^2)^{-\frac{m+n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_1^m (X_i-\mu_1)^2 - \frac{1}{2\sigma^2}\sum_{m+1}^{m+n} (X_i-\mu_2)^2\right\}$$

$$= (2\pi\sigma^2)^{-\frac{m+n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_1^{m+n} X_i^2 + \frac{\mu_1}{\sigma^2}\sum_1^m X_i + \frac{\mu_2}{\sigma^2}\sum_{m+1}^{m+n} X_i - \frac{m\mu_1^2 + n\mu_2^2}{2\sigma^2}\right\}$$

Firstly, note that

$$l(\mu_1, \mu_2, \sigma^2 ; X) = -\frac{m+n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_1^m (X_i-\mu_1)^2 - \frac{1}{2\sigma^2}\sum_{m+1}^{n+m} (X_i-\mu_2)^2$$

$$\frac{\partial l}{\partial \mu_1} = + \frac{\sum_1^m (X_i-\mu_1)}{\sigma^2} \qquad \frac{\partial l}{\partial \mu_2} = + \frac{\sum_{m+1}^{m+n} (X_i-\mu_2)}{\sigma^2}$$

$$\frac{\partial^2 l}{\partial \mu_1 \partial \mu_2} = 0 \qquad \frac{\partial^2 l}{\partial \mu_1^2} = -\frac{m}{\sigma^2} \qquad \frac{\partial^2 l}{\partial \mu_2^2} = -\frac{n}{\sigma^2}$$

Therefore the Hessian is -ve definite and so

$$\hat{\mu}_1 = \frac{\sum_1^m X_i}{m} \qquad \hat{\mu}_2 = \frac{\sum_{m+1}^{m+n} X_i}{n}$$ is the unconstrained MLE.

Therefore if $\frac{1}{m}\sum_1^m X_i \le \frac{1}{n}\sum_{m+1}^{m+n} X_i$, then

this is also the MLE (as the likelihood attains its (unconstrained) maximum, and the constraint is satisfied).

Suppose, on the other hand, that $\frac{1}{m}\sum_{1}^{m} x_i > \frac{1}{n}\sum_{m+1}^{m+n} X_i$

Then the ~~constraint global~~ un. optimum of the likelihood func

is outside the feasible set, and so the constraint $\mu_1 \leq \mu_2$

$$\mathcal{L} = \sum_{1}^{m}(X_i-\mu_1)^2 + \sum_{m+1}^{m+n}(X_i-\mu_2)^2 - \lambda(\mu_1-\mu_2)$$; by KKT, either ~~or~~ $\lambda>0$ or constraint is satisfied ($\mu_1 \leq \mu_2$)

is "biting". By KKT conditions, it suffices to solve

$$\underset{\mu_1,\mu_2}{\text{maximize}} \quad l(\mu_1,\mu_2,\sigma^2;X) \quad \text{s.t.} \quad \mu_1=\mu_2$$

and stationarity implies that, at an optimum,

$$\begin{pmatrix} \frac{1}{\sigma^2}\sum_{1}^{m}(X_i-\mu_1) \\ \frac{1}{\sigma^2}\sum_{m+1}^{m+n}(X_i-\mu_2) \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

~~$\therefore \mu_1 = \frac{1}{m}\left[\sum_{i=1}^{m}(X_i-\lambda)\right] \quad \mu_2 = \frac{1}{n}\left[\sum_{i=m+1}^{m+n}(X_i+\lambda)\right]$~~

~~and $\mu_1=\mu_2 \Rightarrow n\left(\sum_{1}^{m}X_i - m\lambda\right) = m\left(\sum_{m+1}^{m+n}X_j + n\lambda\right)$~~

~~$\Rightarrow 2mn\lambda = n\sum_{1}^{m}X_i - m\sum_{m+1}^{m+n}X_i$~~

~~$\Rightarrow \lambda = \frac{1}{2}\left[\frac{1}{m}\sum_{1}^{m}X_i - \frac{1}{n}\sum_{m+1}^{m+n}X_i\right]$~~

~~$\therefore \hat{\mu}_1 = \frac{1}{m}\sum_{1}^{m}X_i - \lambda \qquad \hat{\mu}_2 \quad \hat{\mu}_2 = \frac{1}{n}\sum_{m+1}^{m+n}X_i + \lambda$~~

~~$= \frac{1}{2}\frac{1}{m}\sum_{1}^{m}X_i + \frac{1}{2}\frac{1}{n}\sum_{m+1}^{m+n}X_i$~~

2011 Q3

$$\therefore \mu_1 = \frac{1}{m}\left[\sum_1^m X_i - \sigma^2\lambda\right] \quad \mu_2 = \frac{1}{n}\left[\sum_{m+1}^{m+n} X_i + \sigma^2\lambda\right]$$

and then $\mu_1 = \mu_2 \implies n\left[\sum_1^m X_i - \sigma^2\lambda\right] = m\left(\sum_{m+1}^{m+n} X_i + \sigma^2\lambda\right)$

$$\implies n\sum_1^m X_i - m\sum_{m+1}^{m+n} X_i = (m+n)\sigma^2\lambda$$

$$\implies \lambda = \frac{n\sum_1^m X_i - m\sum_{m+1}^{m+n} X_i}{(m+n)\sigma^2}$$

$$\therefore \hat{\mu}_1 = \frac{1}{m}\left[\sum_1^m X_i - \frac{n\sum_1^m X_i - m\sum_{m+1}^{m+n} X_i}{m+n}\right]$$

$$= \frac{\sum_1^m X_i + \sum_{m+1}^{m+n} X_i}{m+n}$$

$$= \hat{\mu}_2.$$

Note this can be rewritten ~~$(\hat{\mu}_1, \hat{\mu}_2) = \hat{\mu}_1 = \hat{\mu}_2 = \frac{m}{m+n}\left(\frac{1}{m}\sum_1 X_i\right) +$~~

$$\hat{\mu}_1 = \hat{\mu}_2 = \frac{m}{m+n}\left(\frac{1}{m}\sum_1^m X_i\right) + \frac{n}{m+n}\left(\frac{1}{n}\sum_{m+1}^{m+n} X_i\right) \quad \text{(weighted avg)}$$

Thus the MLE is:

$$(\hat{\mu}_1, \hat{\mu}_2) = \begin{cases} \left(\frac{1}{m}\sum_1^m X_i, \frac{1}{n}\sum_{m+1}^{m+n} X_i\right) & \text{if } \frac{1}{m}\sum_1^m X_i \le \frac{1}{n}\sum_{m+1}^{m+n} X_i \\ \left(\frac{\sum_{i=1}^{m+n} X_i}{m+n}, \frac{\sum_{i=1}^{m+n} X_i}{m+n}\right) & \text{o/w} \end{cases}$$

Ⓘ $= \left(\frac{1}{m}\sum_1^m X_i - \frac{n}{m+n}\left(\frac{1}{m}\sum_1^m X_i - \frac{1}{n}\sum_{m+1}^{m+n} X_i\right)_+ , \frac{1}{n}\sum_{m+1}^{m+n} X_i + \frac{m}{m+n}\left(\frac{1}{m}\sum_1^m X_i - \frac{1}{n}\sum_{m+1}^{m+n} X_i\right)_+\right.$

~~By~~ Asymptotic distn of MLE :

Case 1: $\mu_1 < \mu_2$. Then $\underset{\downarrow p \atop \mu_1}{\frac{1}{m}\sum_1^m X_j} < \underset{\downarrow p \atop \mu_2}{\frac{1}{n}\sum_{m+1}^{m+n} X_j}$ w.h.p. (by WLLN)

$\left(P(A_n) \xrightarrow{n\to\infty} 1 \Rightarrow |P(X_n \le t | A_n) - P(X_n \le t)| \xrightarrow{n\to\infty} 0 \right)$

asymptotic distn by CLT is

$$\left(\sqrt{m}\left(\frac{1}{m}\sum_1^m X_i - \mu_1 \right), \sqrt{n}\left(\frac{1}{n}\sum_{m+1}^{m+n} X_i - \mu_2 \right) \right) \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \right)$$

~~Case 2~~

If we assume $\frac{m}{m+n} = \lambda$ is fixed, then

$$\sqrt{m+n}\left(\left(\frac{1}{m}\sum_1^m X_j, \frac{1}{n}\sum_{m+1}^{m+n} X_j \right) - (\mu_1, \mu_2) \right) \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2/\lambda & 0 \\ 0 & \sigma^2/1-\lambda \end{pmatrix} \right)$$

Case 2: $\mu_1 = \mu_2$

$$\sqrt{N}\left(\begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right) = \begin{cases} \sqrt{N}\begin{pmatrix} \bar{X} - \mu \\ \bar{Y} - \mu \end{pmatrix} & \text{if } \bar{X} \le \bar{Y} \\ \sqrt{N}\begin{pmatrix} \frac{n}{n+m}(\bar{Y} - \mu_2) + \frac{m}{n+m}(\bar{X} - \mu_1) \\ '' \end{pmatrix} & \text{if } \bar{X} > \bar{Y} \end{cases}$$

Write $\bar{\mu}_1 = \frac{1}{m}\sum_1^m X_j$, $\bar{\mu}_2 = \frac{1}{n}\sum_{m+1}^{m+n} X_j$

Assume $m, n \to \infty$ and $\frac{m}{m+n} = \delta$ is fixed.

Then $\sqrt{m}(\bar{\mu}_1 - \mu_1) \xrightarrow{d} N(0, \sigma^2)$, $\sqrt{n}(\bar{\mu}_2 - \mu_2) \xrightarrow{d} N(0, \sigma^2)$ (CLT)

they do so independently $\therefore$

(II) $$\sqrt{N}\left(\begin{pmatrix} \bar{\mu}_1 \\ \bar{\mu}_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right) \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\delta}\sigma^2 & 0 \\ 0 & \frac{1}{1-\delta}\sigma^2 \end{pmatrix} \right)$$

201 Q3

From I note that

$$\cancel{\tilde{\mu}_1 \rightarrow \hat{\mu}_2} \quad \begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix} = \begin{pmatrix} \bar{\mu}_1 - (1-\delta)\,(\bar{\mu}_1 - \bar{\mu}_2)_+ \\ \bar{\mu}_2 + \delta\,(\bar{\mu}_1 - \bar{\mu}_2)_+ \end{pmatrix}$$

If $\mu_1 < \mu_2$, $\bar{\mu}_1 < \bar{\mu}_2$ w.h.p. and so $\begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix} = \begin{pmatrix} \bar{\mu}_1 \\ \bar{\mu}_2 \end{pmatrix}$ w.h.p.

$$\therefore \sqrt{N} \begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix} \xrightarrow{d} N\left(0, \begin{pmatrix} \frac{1}{\delta}\sigma^2 & 0 \\ 0 & \frac{1}{1-\delta}\sigma^2 \end{pmatrix}\right) \quad \text{by [illegible]}$$

If $\mu_1 = \mu_2$, then II still holds, but now

$$\sqrt{N}\left(\begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}\right) = \begin{pmatrix} \sqrt{N}(\bar{\mu}_1 - \mu_1) - \sqrt{N}(1-\delta)(\bar{\mu}_1 - \bar{\mu}_2)_+ \\ \sqrt{N}(\bar{\mu}_2 - \mu_2) - \sqrt{N}\,\delta\,(\bar{\mu}_1 - \bar{\mu}_2)_+ \end{pmatrix}$$

$$\xrightarrow{d} \begin{pmatrix} z_1 - (1-\delta)(z_1 - z_2)_+ \\ z_2 - \delta\,(z_1 - z_2)_+ \end{pmatrix}$$

$$\text{where } \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \sim N\left(0, \begin{pmatrix} \frac{1}{\delta}\sigma^2 & 0 \\ 0 & \frac{1}{1-\delta}\sigma^2 \end{pmatrix}\right)$$

i.e. the limiting distn is

$$\begin{cases} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \text{ where } \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \sim N \text{ and [illegible] } z_1 < z_2 & \text{w.p. } \frac{1}{2} \\ \underbrace{\begin{pmatrix} \delta z_1 + (1-\delta) z_2 \\ \delta z_1 + (1-\delta) z_2 \end{pmatrix}}_{\begin{pmatrix} z \\ z \end{pmatrix} \text{ where } z \sim N(0,\sigma^2)} & \text{w.p. } \frac{1}{2} \end{cases}$$

2011 Q4

(a) We compute the risk directly:

$$R(\theta, \delta_0) = E\left(\mu_1 - \frac{X + \sqrt{n}/2}{n + \sqrt{n}}\right)^2$$

$$= \text{Bias}(\delta_0)^2 + \text{Var}(\delta_0)$$

$$= \left(\mu_1 - \frac{n\mu_1 + \sqrt{n}/2}{n+\sqrt{n}}\right)^2 + \frac{\text{Var}(X)}{(n+\sqrt{n})^2}$$

$$= \left(\frac{\sqrt{n}\,\mu_1 - \sqrt{n}/2}{n+\sqrt{n}}\right)^2 + \frac{n\text{Var}(X_1)}{(n+\sqrt{n})^2}$$

$$= \frac{n\mu_1^2 - n\mu_1 + n/4 + n(\mu_2 - \mu_1^2)}{(n+\sqrt{n})^2}$$

$$= \frac{n(\mu_2 - \mu_1 + 1/4)}{(n+\sqrt{n})^2}$$

This is maximized iff $\mu_2 - \mu_1 = \max_\theta E X_1^2 - E X_1$

But $E X_1^2 - E X_1 = E(X-1)X$ and $X - 1 \leq 0$, $X \geq 0$,

so $(X-1)X \leq 0$. So $E X^2 - E X \leq 0$ and this

quantity is maximized iff $E\,X(X-1) = 0$ ~~ie~~

iff $X(X-1) = 0$ a.s. iff $X = 0$ or 1 a.s. □.

(b) We show δ_0 is minimax.

First, consider the reduced parameter space

Θ_0, the set of Bernoulli distributions on $\{0,1\}$ with success probability p. If p has a Beta$(\sqrt{n}/2, \sqrt{n}/2)$ prior, then δ_0 is a unique Bayes estimator, with constant risk $\frac{n/4}{(n+\sqrt{n})^2}$ (by class results).

$\therefore$ δ_0 is minimax on Θ_0.

But from part (i),

$$\sup_{\Theta_0} R(\theta,\delta_0) = \sup_{\Theta} R(\theta,\delta_0)$$

$\therefore$ δ_0 is minimax for $\theta \in \Theta$. $\square$

2009 Q1

(i) MLE for λ is $\bar{X}$.

∴ MLE for $g(\lambda)$ is $g(\bar{X})$

∴ MLE is $\bar{X}e^{-\bar{X}}$.

(ii) By CLT, $\sqrt{n}(\bar{X}-\lambda) \xrightarrow{d} N(0,\lambda)$

By Δ-method, $\sqrt{n}(g(\bar{x})-g(\hat{\lambda})) \xrightarrow{d} N(0, g'(\lambda)^2\lambda)$

$g(\lambda) = \lambda e^{-\lambda}$ ∴ $g'(\lambda) = e^{-\lambda} - \lambda e^{-\lambda}$

∴ $\sqrt{n}(\bar{x}e^{-\bar{x}} - \lambda e^{-\lambda}) \xrightarrow{d} N(0, e^{-2\lambda}(1-\lambda)^2\lambda)$

i.e. $\sqrt{n}(\hat{g}_n - \lambda e^{-\lambda}) \xrightarrow{d} N(0, e^{-2\lambda}(1-\lambda)^2\lambda)$

On the other hand, $n\hat{p}_n \sim B(n, \lambda e^{-\lambda})$.

∴ $\sqrt{n}(\hat{p}_n - \lambda) \xrightarrow{d} N(0, \lambda e^{-\lambda}(1-\lambda e^{-\lambda}))$

∴ Asymptotic relative efficiency is:

$$\frac{1/\text{Var}\,\hat{p}_n}{1/\text{Var}\,\hat{g}_n} = \frac{1/\lambda e^{-\lambda}(1-\lambda e^{-\lambda})}{1/e^{-2\lambda}(1-\lambda)^2\lambda} = \cancel{\frac{1-\lambda e^{-\lambda}}{e^{-\lambda}(1-\lambda)^2}} \quad \frac{e^{-\lambda}(1-\lambda)^2}{1-\lambda e^{-\lambda}}$$

(iii) Compute $g''(\lambda) = -e^{-\lambda} - e^{-\lambda} + \lambda e^{-\lambda} = (\lambda-2)e^{-\lambda}$

Thus, in the case $\lambda = 1$, as $g'(\lambda) = 0$, we have

by the modified Δ-method,

$n(\hat{q}_n - \lambda e^{-\lambda}) \xrightarrow[\lambda=1]{d}$ ~~$N(0, g''(\lambda)\sigma^2)$~~ $\frac{1}{2} g''(\lambda) \lambda \chi_1^2$

$\therefore \quad n(\hat{q}_n - e^{-1}) \xrightarrow[\lambda=1]{d} -\frac{e^{-1}}{2} \chi_1^2$

or $\quad n\left(\bar{X} e^{-\bar{X}} - \frac{1}{e}\right) \xrightarrow[\lambda=1]{d} -\frac{1}{2e} \chi_1^2$

2009 Q3

(i) The Bayes estimator is the posterior mean.

$$\pi(\theta|x) \propto L(\theta;x)\,\pi(\theta)$$

$$L(\theta;x) = \prod_{i=1}^{n} e^{-(Y_i-\theta)}\,\mathbb{1}\{Y_i>\theta\}$$

$$= e^{-(\sum Y_i - n\theta)}\,\mathbb{1}\{Y_{(1)}>\theta\}$$

$$\therefore\ \pi(\theta|Y) \propto e^{-(\sum Y_i - n\theta)}\,\mathbb{1}\{\theta<Y_{(1)}\}\,e^{-\theta}\,\mathbb{1}\{\theta>0\}$$

$$\propto e^{(n-1)\theta}\,\mathbb{1}\{0<\theta<Y_{(1)}\}$$

$\therefore$ But $\int_0^{Y_{(1)}} e^{(n-1)\theta}\,d\theta = \left[\frac{1}{n-1}e^{(n-1)\theta}\right]_0^{Y_{(1)}}$

$$= \frac{e^{(n-1)Y_{(1)}}-1}{n-1}$$

$$\therefore\ \pi(\theta|Y) = \frac{n-1}{e^{(n-1)Y_{(1)}}-1}\,e^{(n-1)\theta}\,\mathbb{1}\{0<\theta<Y_{(1)}\}$$

$$\therefore\ \delta_n^{\pi} = E\,\theta|Y = \int_0^{Y_{(1)}} \frac{n-1}{e^{(n-1)Y_{(1)}}-1}\,\theta\, e^{(n-1)\theta}\,d\theta$$

$$= \frac{n-1}{e^{(n-1)Y_{(1)}}-1}\left[\left(\theta\,\frac{e^{(n-1)\theta}}{n-1}\right)_0^{Y_{(1)}} - \int_0^{Y_{(1)}} \frac{e^{(n-1)\theta}}{n-1}\,d\theta\right]$$

$$= \frac{n-1}{e^{(n-1)Y_{(1)}}-1}\left[\frac{Y_{(1)}\,e^{(n-1)Y_{(1)}}}{n-1} - \frac{e^{(n-1)Y_{(1)}}-1}{(n-1)^2}\right]$$

$$= \frac{(Y_{(1)}(n-1)-1)\,e^{(n-1)Y_{(1)}}+1}{(n-1)(e^{(n-1)Y_{(1)}}-1)}$$

$$= \frac{Y_{(1)}(n-1)e^{(n-1)Y_{(1)}} - e^{(n-1)Y_{(1)}} + 1}{(n-1)e^{(n-1)Y_{(1)}} - (n-1)}$$

$$= Y_{(1)} + \frac{(n-1) - e^{(n-1)Y_{(1)}} + 1}{(n-1)e^{(n-1)Y_{(1)}} - (n-1)}$$

$$= Y_{(1)} - \frac{1}{n-1} + \frac{n-1}{(n-1)e^{(n-1)Y_{(1)}} - (n-1)}$$

$$= Y_{(1)} - \frac{1}{n-1} + \frac{1}{e^{(n-1)Y_{(1)}} - 1}$$

(ii) Suppose $\theta_0 > 0$ is the true parameter

Then $S_n^{\pi} - \theta_0 = (Y_{(1)} - \theta_0) + \frac{1}{e^{(n-1)Y_{(1)}} - 1} - \frac{1}{n-1}$

and $Y_{(1)} - \theta_0 \overset{d}{=} Z_{(1)}$ where $Z_1, \dots, Z_n \overset{iid}{\sim} Exp(1)$

Now compute $P(d_n (Z_{(1)}) > t) = P(Z_{(1)} > \frac{t}{d_n})$

$$= P(Z_i > \frac{t}{d_n} \quad \forall i)$$

$$= (e^{-\frac{t}{d_n}})^n \qquad \text{(independence)}$$

$$= e^{-t} \quad \text{if we pick } d_n = n.$$

Thus $n(Y_{(1)} - \theta_0) \overset{d}{=} Exp(1)$

Note, moreover, that $\frac{n}{n-1} \to 1$ and

2009 Q3 Bernstein von Mises theorem

$\Rightarrow$ expect similar convergence to MLE

$$0 \le \frac{n}{e^{(n-1)Y_{(1)}}-1} \le \frac{n}{(n-1)Y_{(1)}+(n-1)^2Y_{(1)}^2/2} \xrightarrow{P} 0$$

Since $n(Y_{(1)}-\theta_0)$ ~~[illegible]~~ converges in distn,

so $Y_{(1)}-\theta_0 \xrightarrow{P} 0$ so $Y_{(1)} \xrightarrow{P} \theta_0 > 0$

so that $$\frac{n}{(n-1)Y_{(1)}+\frac{(n-1)^2Y_{(1)}^2}{2}} = \frac{n/(n-1)^2}{\frac{Y_{(1)}}{n-1}+\frac{1}{2}Y_{(1)}} \xrightarrow{P} \frac{0}{0+\theta_0} = 0$$

by Slutsky's thm.

Hence,

$$n(\delta_n^\pi - \theta_0) = n(Y_{(1)}-\theta_0) + \frac{n}{e^{(n-1)Y_{(1)}}-1} - \frac{n}{n-1}$$

$$\xrightarrow{d} -1 + \text{Exp}(1) \quad \text{by Slutsky's.}$$

~~for large n, $P(n(\delta_n^\pi - \theta_0)$~~

But

$$P(-\log(1-\tfrac{\alpha}{2}) < \text{Exp}(1) < -\log\tfrac{\alpha}{2}) = e^{+\log(1-\frac{\alpha}{2})} - e^{+\log(\frac{\alpha}{2})}$$

$$= 1-\tfrac{\alpha}{2}-\tfrac{\alpha}{2} = 1-\alpha.$$

Hence, for large n,

$$P\left(-1-\log(1-\tfrac{\alpha}{2}) < n(\delta_n^\pi-\theta_0) < -1-\log\tfrac{\alpha}{2}\right) \approx 1-\alpha$$

$\therefore$ $1-\alpha$ asymptotic C.I. is $\theta_0 \in \left(\delta_n^\pi + \frac{1+\log\frac{\alpha}{2}}{n}, \delta_n^\pi + \frac{1+\log(1-\frac{\alpha}{2})}{n}\right)$ $\square$

2008 Q1

(i) $L(\theta;X) = \theta^{-n} \mathbb{1}\{\theta < X_{(1)}\} \mathbb{1}\{X_{(n)} < 2\theta\}$

$$\therefore \frac{L(\theta;X)}{L(\theta;Y)} = \frac{\mathbb{1}\{\theta < X_{(1)}\} \mathbb{1}\{X_{(n)} < 2\theta\}}{\mathbb{1}\{\theta < Y_{(1)}\} \mathbb{1}\{Y_{(n)} < 2\theta\}}$$

this is independent of θ iff $(X_{(1)}, X_{(n)}) = (Y_{(1)}, Y_{(n)})$

$\therefore \quad T(X) = (X_{(1)}, X_{(n)})$ is M.S.

(ii) $X_1, \dots, X_n \overset{d}{=} \theta + \theta U_1, \dots, \theta + \theta U_2$ where $U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$.

$\therefore \quad X_{(1)}, \dots, X_{(n)} \overset{d}{=} \theta + \theta U_{(1)}, \dots, \theta + \theta U_{(n)}$

$$\therefore \frac{X_{(n)}}{X_{(1)}} \overset{d}{=} \frac{1 + U_{(n)}}{1 + U_{(1)}} \quad \square$$

(iii) $\sup_\theta L(\theta;X)$? to maximise likelihood, pick smallest θ

s.t. $\theta \le X_{(1)}$ and $2\theta \ge X_{(n)}$ ~~$\therefore \hat{\theta} = \min\left(X_{(1)}, \frac{X_{(n)}}{2}\right)$~~

$\therefore \quad \hat{\theta} = \frac{X_{(n)}}{2}$

~~$\therefore \Lambda(X_1, \dots, X_n) = \frac{\theta_0^{-n} \mathbb{1}\{\theta_0 \le X_{(1)}\} \mathbb{1}\{2\theta_0 \ge X_{(n)}\}}{\left(\frac{X_{(n)}}{2}\right)^{-n} \mathbb{1}\{\frac{X_{(n)}}{2} \le X_{(1)}\}}$~~

$\therefore -2\log\Lambda(X_1, \dots, X_n) = 2\log\left(\frac{X_{(n)}}{2}\right)^{-n} \mathbb{1}\{\frac{X_{(n)}}{2} \le X_{(1)}\} \mathbb{1}\{X_{(n)} \ge X_{(1)}\} - 2\log\theta_0^{-n}$

$= 2n\log\theta_0 - 2n\log X_{(n)} + 2n\log 2$ if $\theta_0 < X_{(1)} < X_{(n)} < 2\theta_0$

and $\underbrace{X_{(1)} \ge \frac{X_{(n)}}{2}}$

$= +\infty$ if $X_{(n)} < \theta_0$ or $X_{(n)} > 2\theta_0$

LRT rejects if

$-2\log\Lambda > k$ where k is s.t.

$P_{\theta_0}(-2\log\Lambda > k) = \alpha$

$\therefore P_{\theta_0}(2n\log\theta_0 + 2n\log 2 - 2n\log X_{(n)} > k) = \alpha$

$\therefore P_{\theta_0}\left(\log X_{(n)} < \log(2\theta_0) - \frac{k}{2n}\right) = \alpha$

$\therefore P_{\theta_0}\left(X_{(n)} < 2\theta_0 e^{-\frac{k}{2n}}\right) = \alpha$ (I)

$\therefore \left(\frac{2\theta_0 e^{-\frac{k}{2n}} - \theta_0}{\theta_0}\right)^n = \alpha \quad \therefore \left(2e^{-\frac{k}{2n}} - 1\right)^n = \alpha$

$\therefore 2e^{-\frac{k}{2n}} - 1 = \alpha^{1/n} \quad \therefore 2e^{-\frac{k}{2n}} = 1 + \alpha^{1/n}$

$\therefore -\frac{k}{2n} = \log\frac{1+\alpha^{1/n}}{2} \quad \therefore \boxed{k = -2n\log\frac{1+\alpha^{1/n}}{2}}$

using $\alpha = 0.05$ gives the desired answer, test is $\phi = 1$ if $X_{(n)} < \theta_0(1+\alpha^{\frac{1}{n}})$

(or if $X_{(n)} < \theta_0$ or ~~$X_{(n)}$~~ $X_{(n)} \geq 2\theta_0$.

~~(iv) From I, our 95% confidence set for θ is~~

~~$\theta \in \left(\frac{X_{(n)}}{2e^{-\frac{k}{2n}}} < \theta_0\right)$~~

~~$\theta \in \left(\frac{X_{(n)}}{1+\alpha^{1/n}}, \infty\right)$ is a 95% C.I.~~

2008 Q1

(iv) From I,

$$P_{\theta_0}\left(X_{(n)} > 2\theta_0 e^{-\frac{k}{2n}} \right) = 1-\alpha$$

and noting that $2e^{-\frac{k}{2n}} = 1+\alpha^{\frac{1}{n}}$,

$\theta_0 < \frac{X_{(n)}}{1+\alpha^{1/n}}$ is a $1-\alpha$ confidence region.

Note that we can construct a smaller confidence region:

$$P_\theta\left(X_{(n)} > \theta_0 (1+\alpha^{\frac{1}{n}}) \right) = P_\theta\left(X_{(n)} > \theta (1+\alpha^{\frac{1}{n}}) \text{ and } X_{(n)} < 2\theta \text{ and } X_{(1)} > \theta \right)$$

$$\therefore \quad \theta \in \left(\frac{X_{(n)}}{2}, \underbrace{\frac{X_{(n)}}{1+\alpha^{1/n}}}_{\min\{X_{(1)}, \frac{X_{(n)}}{1+\alpha^{1/n}}\}} \right) \text{ is a } 1-\alpha \text{ C.I.} \quad \square$$

(v) $P_\theta\left(X_{(n)} > \theta(1+\alpha^{\frac{1}{n}}) \,\middle|\, X_{(1)}/X_{(n)} = 1/2 \right)$

$$= P_\theta\left(2X_{(1)} > \theta(1+\alpha^{\frac{1}{n}}) \,\middle|\, X_{(1)}/X_{(n)} = 1/2 \right)$$

$$= P_\theta\left(X_{(1)} > \theta\left(\frac{1+\alpha^{\frac{1}{n}}}{2}\right) \,\middle|\, '' \right)$$

$$\geq P_\theta\left(X_{(1)} > \theta \,\middle|\, '' \right) = 1$$

Secondly,

$P_\theta\left(X_{(n)} > \theta(1+\alpha^{\frac{1}{n}}) \,\middle|\, X_{(1)}/X_{(n)} = 1 \right)$?

$= P(U(\theta, 2\theta) > \theta(1+\alpha^{1/n})) = 1-\alpha^{1/n}$ by symmetry

~~$= P_\theta(\;= P_\theta((X_{(n)}-\theta) > \theta\alpha^{1/n} \mid X_{(1)}/X_{(n)} = 1)$~~

$\left(= P_\theta(X_{(n)} > \theta(1+\alpha^{\frac{1}{n}}) \mid$ ~~$X_0 = X_0$~~ $\forall i\; X_i = X_{(1)}\; \forall i) = P_\theta(X_1 > \theta(1+\alpha^{\frac{1}{n}}) \mid \text{all } X_i \text{ are equal}) \right)$

~~$= P_\theta((X_{(n)}-\theta) > \theta\alpha^{1/n})$ — — by Bayes'~~

~~Compute $f_{X_{(1)},X_{(n)}}(x_1,x_n) = \frac{n!}{(n-2)!}\cdot\frac{1}{\theta}\left(\frac{x_n-x_1}{\theta}\right)^{n-2}$~~

~~and f~~ ~~(as $X_{(n)}-\theta$ [illegible] is the nth order statistic of a $U(0,\theta)$ family so it is C.I. for θ, whereas $\frac{X_{(1)}}{X_{(n)}}$ is ancillary)~~

~~$= P\left(\frac{X_{(n)}-\theta}{\theta} > \alpha^{1/n}\right) = P$~~

~~$= P\left(U_{(n)} > \alpha^{1/n}\right) =$~~

~~$= 1 - \alpha^{\frac{1}{n}}$~~

∴ It is an appropriate C.I.

~~However~~ Clearly if $X_{(1)}/X_{(n)} = \frac{1}{2}$, we know exactly what the parameter θ should be, so it makes sense that our C.I. has 100% coverage

On the other hand, if $X_{(1)}/X_{(n)} = 1$, we have the least possible information about the spread; so the confidence level $1-\alpha^{\frac{1}{n}}$ is the smallest and $\downarrow 0$ as $n \to \infty$

(as our interval becomes narrower while we get no additional information)

2008 Q2

(i) $L(\rho;\vec{x},\vec{y}) = \prod_{1}^{n} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[x_i^2 - 2\rho x_i y_i + y_i^2\right]\right\}$

$= \frac{1}{(\sqrt{1-\rho^2}\,2\pi)^n} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\sum x_i^2 - 2\rho \sum x_i y_i + \sum y_i^2\right]\right\}$

fix $\rho_0 = 0,\ \rho_1 = \frac{1}{2}$ ~~then~~ let

$T = \frac{L(\rho_1;\vec{x},\vec{y})}{L(\rho_0;\vec{x},\vec{y})} = \cancel{\exp\{-\frac{1}{2(1-\rho^2)}}$

$= \left(\frac{4}{3}\right)^{\frac{n}{2}} \exp\left\{\frac{1}{2}\left(\sum x_i^2 + \sum y_i^2\right) - \frac{2}{3}\left(\sum x_i^2 - \sum x_i y_i + \sum y_i^2\right)\right\}$

$=$

$\frac{L(\rho;\vec{x},\vec{y})}{L(\rho;\tilde{x},\tilde{y})} = \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\overbrace{\sum x_i^2 - \sum \tilde{x}_i^2}^{I} - 2\rho\overbrace{\left(\sum x_i y_i - \sum \tilde{x}_i \tilde{y}_i\right)}^{II} + \overbrace{\sum y_i^2 - \sum \tilde{y}_i^2}^{III}\right]\right\}$

Claim:

$\left(\sum x_i^2 + y_i^2, \sum x_i y_i\right) = \left(\sum \tilde{x}_i^2 + \tilde{y}_i^2, \sum \tilde{x}_i \tilde{y}_i\right)$

this ratio is independent of ρ iff $\left(\sum x_i^2, \sum x_i y_i, \sum y_i^2\right) = \left(\sum \tilde{x}_i^2, \sum \tilde{x}_i \tilde{y}_i, \sum \tilde{y}_i^2\right)$

Proof: $\Leftarrow$: obvious.

$\Rightarrow$: if $\sum x_i y_i \neq \sum \tilde{x}_i \tilde{y}_i$ then the II term has a ~~power of ρ~~ non-zero coefficient $\frac{\rho}{1-\rho^2}$

next, if ~~$\sum x_i^2 \neq \sum \tilde{x}_i^2$~~, $\sum x_i^2 + y_i^2 \neq \sum \tilde{x}_i^2 + \tilde{y}_i^2$, then we have a $\frac{1}{1-\rho}$ term with non-zero coefficient

$\therefore \boxed{T = \left(\sum x_i^2 + y_i^2, \sum x_i y_i\right) \text{ is M.S.}}$

$E f(T) = E(T_1 - 2n) = \sum 2n - 2n = 0 \quad \forall \rho \quad \therefore$ T not C.S.

(ii) $f_\theta(x) = g_\theta(T(x))\, h(x)$

Therefore MLE must be a function of T ← sufficient statistic

Now note that r_n is invariant to shifts in the mean, whereas T_1, T_2 is not

Suppose r_n is sufficient. ~~then~~
Then $(T_1, T_2) = g(r_n(X))$ for some g.
But $r_n(X)$ is invariant to shifts in X, (or scalings in X)
whereas (T_1, T_2) is not.
Therefore there cannot be such g.

2008 Q3

(i) $P(Y=y) = \theta^{y-1}(1-\theta)$ if $y = 1, 2, \ldots, c$

$\qquad\qquad\quad \theta^{c}$ if $y = c+1$

As $Y \geq 0$, we know that

$$EY = \sum_{y=0}^{\infty} P(Y > y)$$

but if $y \leq c$, $P(Y>y) = P(X>y) = \sum_{y+1}^{\infty} \theta^{i-1}(1-\theta) = \theta^{y}$

and if $y \geq c+1$, $P(Y>y) = 0$ (it is censored)

$$\therefore \quad EY = \sum_{y=0}^{\infty} \theta^{y} \mathbb{1}\{y \leq c\} = \sum_{k=0}^{c} \theta^{k} \qquad \square$$

(ii) $$L(\theta;Y) = \prod_{i=1}^{n} \left\{\theta^{Y_i - 1}(1-\theta)\right\}^{\mathbb{1}\{Y_i \leq c\}} \left\{\theta^{c}\right\}^{\mathbb{1}\{Y_i = c+1\}}$$

$$= (\theta^{c})^{R} (1-\theta)^{n-R} \prod_{i=1}^{n} \theta^{Y_i \mathbb{1}\{Y_i \leq c\} - \mathbb{1}\{Y_i \leq c\}}$$

$$= \theta^{Rc} (1-\theta)^{n-R} \theta^{(\Sigma Y_i - (c+1)R)} \theta^{-(n-R)}$$

~~$= \theta^{\Sigma Y_i - R}$~~

$$= \theta^{Rc + \Sigma Y_i - cR - R - n + R} (1-\theta)^{n-R}$$

$$= \theta^{\Sigma Y_i - n} (1-\theta)^{n-R}$$

$\therefore$ $\ell(\theta;Y) = (\Sigma Y_i - n)\log\theta + (n-R)\log(1-\theta)$

$\ell'(\theta;Y) = \frac{\Sigma Y_i - n}{\theta} - \frac{n-R}{1-\theta}$

$\ell''(\theta;Y) = -\frac{\Sigma Y_i - n}{\theta^2} - \frac{n-R}{(1-\theta)^2} < 0$ since $\Sigma Y_i \geq n$ (as $Y_i \geq 1 \ \forall i$) and $R \leq n$

and if $R = n$ then $\sum Y_i > n$.

Hence the ℓ is strictly concave and setting $\ell'(\theta; Y) = 0$ gives the MLE

$$\therefore \quad \frac{\sum Y_i - n}{\theta} - \frac{n - R}{1-\theta} = 0$$

$$\therefore \quad (\sum Y_i - n) = (n - R)\theta + (\sum Y_i - n)\theta$$

$$\therefore \quad \hat{\theta}_{MLE} = \frac{\sum Y_i - n}{\sum Y_i - R} \quad \square .$$

(iii) $$\hat{\theta}_{MLE} = \frac{\frac{1}{n}\sum Y_i - 1}{\frac{1}{n}\sum Y_i - \frac{1}{n}R}$$

Now by WLLN, $\frac{1}{n}\sum Y_i \xrightarrow{P} EY = \sum_{k=0}^{c} \theta^k$

$$\frac{1}{n}R \xrightarrow{P} E\,\mathbb{1}_{\{Y_i = c+1\}} = P(Y = c+1) = \theta^c$$

$\therefore$ By Slutsky,

$$\hat{\theta}_{MLE} \xrightarrow{P} \frac{\sum_{k=0}^{c} \theta^k - 1}{\sum_{k=0}^{c} \theta^k - \theta^c} = \frac{\sum_{k=1}^{c} \theta^k}{\sum_{k=0}^{c-1} \theta^k} = \theta \frac{\sum_{k=0}^{c-1} \theta^k}{\sum_{k=0}^{c-1} \theta^k} = \theta .$$

$\therefore$ $\hat{\theta}$ is consistent $\square$

2008 Q4

(i) $p(\vec{x} \mid \lambda_1, \ldots, \lambda_n) = \prod_{i=1}^{n} \frac{e^{-\lambda_i} \lambda_i^{x_i}}{x_i!}$

$$= \exp\left\{ \sum_i^n x_i \log \lambda_i - \sum \lambda_i \right\} h(X)$$

$$= \exp\left\{ \left(\sum_1^n x_i \right) \log \lambda + \sum_{i=1}^{n} x_i \log\left(\frac{\lambda_i}{\lambda} \right) - \lambda \right\} h(X)$$

~~Reparametrise to $\lambda = \sum_i^n \lambda_i$, $p_1 = \frac{\lambda_1}{\lambda}$, ..., $p_{n-1} = \frac{\lambda_{n-1}}{\lambda}$~~.

~~Then $p(\vec{x} \mid \lambda, p_1, \ldots, p_{n-1}) = \exp\left\{ \left(\sum_1^n x_i \right) \lambda + \sum_1^{n-1} x_i \log p_i + x_n \log\left(1 - \sum_1^{n-1} p_i\right) - \lambda \right\} h(x)$~~

Fix an alternative $\lambda_1', \ldots, \lambda_n'$ s.t. $\lambda' = \sum_1^n \lambda_i' > n$.

As a least favourable prior, put mass 1 on

$$\lambda_i = n \frac{\lambda_i'}{\lambda'} \quad \forall i$$

$\therefore$ MP test for this null against this alternative is (NP lemma)

$$\phi(x) = 1 \quad \text{if} \quad \frac{p(\vec{x} \mid \vec{\lambda}')}{p(\vec{x} \mid \vec{\lambda}_0)} > k$$

$$= 0 \quad \text{o/w} \quad \text{(as we ignore random continuity correction)}$$

$E_{\vec{\lambda}_0} \phi \le \alpha$

Compute $\frac{p(\vec{x} \mid \vec{\lambda}')}{p(\vec{x} \mid \vec{\lambda}_0)} = \exp\left\{ (\sum x_i)(\log \lambda' - \log n) + \sum x_i \left(\log \frac{\lambda_i'}{\lambda'} - \log \frac{\lambda_i}{n} \right) - (\lambda' - n) \right\}$

$$= \exp\left\{ (\sum x_i) \log \frac{\lambda'}{n} - (\lambda' - n) \right\}$$

$\therefore \quad \phi(x) = 1 \quad \text{iff} \quad \sum x_i > k'$

Now under H_0, $\Sigma X_i \sim \text{Poisson}(n)$ so $K' = \text{Poisson}_{n, 1-\alpha}$

is the $1-\alpha$ quantile of a Poisson (n) (as we ignore continuity)

$\therefore \quad \phi(x) = \begin{cases} 1 & \text{if } \Sigma x_i > K' \\ 0 & \text{o/w} \end{cases}$

is MP for this problem.

But ϕ is level α for the original problem

$\therefore$ ϕ is MP for $H_0: \lambda = n$ vs $(\lambda_1', \lambda_2', \ldots, \lambda_n')$

But ϕ is free of the alternative.

$\Rightarrow$ ϕ is UMP for $\lambda = n$ vs $\lambda > n$. $\square$

2007 Q1

(i) $L_{\theta,\mu}(x) = g_{\theta,\mu}(T_1)\, h_{\mu}(x) = \tilde{g}_{\theta,\mu}(T_2)\, \tilde{h}_{\theta}(x)$

$$\frac{L_{\theta,\mu}(x)}{L_{\theta_0,\mu_0}(x)} = \underbrace{\frac{L_{\theta,\mu}(x)}{L_{\theta,\mu_0}(x)}}_{\frac{\tilde{g}_{\theta,\mu}(T_2)\,\cancel{\hat{h}_{\theta}(x)}}{\tilde{g}_{\theta,\mu_0}(T_2)\,\cancel{\tilde{h}_{\theta}(x)}}} \cdot \underbrace{\frac{L_{\theta,\mu_0}(x)}{L_{\theta_0,\mu_0}(x)}}_{\frac{g_{\theta,\mu_0}(T_1)\,\cancel{h_{\mu}(x)}}{g_{\theta_0,\mu}(T_1)\,\cancel{h_{\mu}(x)}}}$$

$$\therefore \quad L_{\theta,\mu}(x) = \left\{ \frac{\tilde{g}_{\theta,\mu}(T_2)}{\tilde{g}_{\theta,\mu_0}(T_2)} \cdot \frac{g_{\theta,\mu_0}(T_1)}{g_{\theta_0,\mu_0}(T_1)} \right\} L_{\theta_0,\mu_0}(x)$$

(ii) $f_{\theta_1,\theta_2}(x) = \tilde{g}_{\theta_1,\theta_2}(T_1)\, h_{\theta_2}(x)$

discrete case $\quad P(T_2 = t_2 \mid T_1 = t_1) = \dfrac{P(T_2 = t_2 \}, T_1 = t_1)}{P(T_1 = t_1)}$

$$= \frac{g_{\theta_1,\theta_2}(t_1) \displaystyle\sum_{x : T_1(x) = t_1,\, T_2(x) = t_2} h_{\theta_2}(x)}{g_{\theta_1,\theta_2}(t_1) \displaystyle\sum_{x : T_1(x) = t_1} h_{\theta_2}(x)}$$

2007 Q4

(i) $p_N(x) = \prod_{i=1}^{n} \binom{N}{x_i} \left(\frac{1}{2}\right)^N$ For $x_i \in \{0, 1, \ldots, N\}$

$\therefore L(N;X) = \prod_{i=1}^{n} \binom{N}{x_i} \left(\frac{1}{2}\right)^N$, $N \in \{X_{(n)}, X_{(n)}+1, X_{(n)}+2, \ldots\}$

$N \geq X_{(n)}$.

Fix $\epsilon \in (0,1)$

$$P(|X_{(n)} - N| > \epsilon) = P(X_{(n)} < N)$$

$$= P(X_i \neq N \ \forall i)$$

$$= \{P(X_i < N)\}^n \qquad \text{(independence)}$$

$$= \left(1 - \left(\tfrac{1}{2}\right)^N\right)^n$$

$$\longrightarrow 0 \text{ as } n \to \infty \qquad \square$$

$\therefore X_{(n)}$ is consistent.

(ii) $L(N;X) = \prod_{i=1}^{n} \binom{N}{x_i} \frac{1}{2^N}$

$$= \frac{1}{2^{nN}} \prod_{i=1}^{n} \binom{N}{X_{(i)}} \quad \text{for } N \geq X_{(n)}$$

If $X_{(n)}$ is an MLE, then $L(X_{(n)};X) \geq L(X_{(n)}+1;X)$

which implies $\frac{1}{2^{nX_{(n)}}} \prod_{i=1}^{n} \binom{X_{(n)}}{X_{(i)}} \geq \frac{1}{2^{n(X_{(n)}+1)}} \prod_{i=1}^{n} \binom{X_{(n)}+1}{X_{(i)}}$

$$\therefore \prod_{i=1}^{n} \binom{X_{(n)}}{X_{(i)}} \Big/ \binom{X_{(n)}+1}{X_{(i)}} \geq 2^{-n}$$

∴ $\cancel{\prod_{i=1}^{n} \frac{x_{(n)}!}{x_i!(x_{(n)}-x_i)!}}$

Proof $\binom{n}{k} \Big/ \binom{n+1}{k} = \frac{n!}{k!(n-k)!} \Big/ \frac{(n+1)!}{k!(n+1-k)!} = \frac{n+1-k}{n+1} = 1 - \frac{k}{n+1}$

∴ $\prod_{i=1}^{n} \left(1 - \frac{x_i}{x_{(n)}+1}\right) \geq 2^{-n}$ □

Conversely, suppose $\prod_{i=1}^{n} \left(1 - \frac{x_i}{x_{(n)}+1}\right) \geq 2^{-n}$

Then, $\cancel{\prod_{i=1}^{n} \left(1 - \frac{x_i}{x_{(n)}+1}\right) \geq 2^{-n}}$ also,

$\prod_{i=1}^{n} \left(1 - \frac{x_i}{x_{(n)}+k}\right) \geq 2^{-n}$, $\forall k \in \mathbb{Z}^+$, [illegible] ①

as $1 - \frac{x_i}{x_{(n)}+1} \leq 1 - \frac{x_i}{x_{(n)}+k}$ since $k \geq 1$.

But $L(N;X) \geq L(N+1;X) \iff \frac{1}{2^{nN}} \prod \binom{N}{x_i} \geq \frac{1}{2^{n(N+1)}} \prod \binom{N+1}{x_i}$

$\iff \prod_{i=1}^{n} \binom{N}{x_i} \Big/ \binom{N+1}{x_i} \geq 2^{-n} \iff \prod_{i=1}^{n} \left(1 - \frac{x_i}{N+1}\right) \geq 2^{-n}$

Therefore, by I, $L(X_{(n)}, X) \geq L(X_{(n)}+1; X) \geq L(X_{(n)}+2; X) \geq \dots$

∴ $L(N;X)$ is maximal at $N = X_{(n)}$ ∴ $X_{(n)}$ is an MLE. □

Note that if we require MLE to be a UNIQUE maximizer, we should use the condition $\prod \left(1 - \frac{x_i}{x_{(n)}+1}\right) > 2^{-n}$ with strict inequality instead.

2007 Q5

(i) $L(\alpha,\beta;X) = \prod_{i=1}^{n} \frac{1}{2}\beta(1-e^{-\beta\alpha})^{-1} e^{-\beta|X_i|} \mathbb{1}\{-\alpha \le X_i \le \alpha\}$

$= 2^{-n}\beta^{n}(1-e^{-\beta\alpha})^{-n} e^{-\beta\sum|X_i|} \mathbb{1}\{-\alpha \le X_{(1)}\}\mathbb{1}\{X_{(n)} \le \alpha\}$

For any fixed β, note that $-\beta\alpha$ is a decreasing in α

so $e^{-\beta\alpha}$ is decreasing in α so $1-e^{-\beta\alpha}$ is increasing

in α so $\frac{1}{1-e^{-\beta\alpha}}$ is decreasing in α and

also $\frac{1}{(1-e^{-\beta\alpha})^{n}} = (1-e^{-\beta\alpha})^{-n}$ is decreasing in α.

Thus, the MLE for α will be the smallest value of

α s.t. $-\alpha \le X_{(1)}$ and $X_{(n)} \le \alpha$ (if these conditions

don't hold, then $L = 0$). Hence

$$\hat{\alpha} = \max_i |X_i| .$$

To find the MLE for β it remains to maximize

$$L(\hat{\alpha},\beta;X) = 2^{-n}\beta^{n}(1-e^{-\beta\hat{\alpha}})^{-n} e^{-\beta\sum|X_i|}$$

Compute $\ell(\hat{\alpha},\beta;X) = n\log\beta - n\log(1-e^{-\beta\hat{\alpha}}) - \beta\sum|X_i|$

$$\therefore \frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - \frac{n}{1-e^{-\beta\hat{\alpha}}}\left(\hat{\alpha}e^{-\beta\hat{\alpha}}\right) - \sum|X_i|$$

$$\therefore \frac{\partial^2 l}{\partial \beta} = -\frac{n}{\beta^2} - n\hat{\alpha}\,\frac{(1-e^{-\beta\hat{\alpha}})(-\hat{\alpha}e^{-\beta\hat{\alpha}}) - e^{-\beta\hat{\alpha}}(\hat{\alpha}e^{-\beta\hat{\alpha}})}{(1-e^{-\beta\hat{\alpha}})^2}$$

$$\frac{\partial^2 l}{\partial \beta^2} = -\frac{n}{\beta^2} + \frac{n\hat{\alpha}^2 e^{-\beta\hat{\alpha}}}{(1-e^{-\beta\hat{\alpha}})^2} < 0$$

$\therefore$ $\hat{\beta}$ is the unique solution to the equation

$$\frac{\partial l}{\partial \beta}(\hat{\alpha},\beta;x) = 0 \quad \text{i.e.} \quad \frac{n}{\beta} - \frac{n\hat{\alpha}e^{-\hat{\alpha}\beta}}{1-e^{-\hat{\alpha}\beta}} - \sum|X_i| = 0$$

(iii) Work for $\hat{\alpha}$ first.

~~$P(n^{\gamma}(\hat{\alpha}-$~~

$$P(n^{\gamma}(\alpha-\hat{\alpha}) \leq t) = P(\hat{\alpha} \geq \alpha - tn^{-\gamma})$$

$$= 1 - P(\max|X_i| < \alpha - tn^{-\gamma})$$

$$= 1 - \left[P(|X_i| < \alpha - tn^{-\gamma})\right]^n \qquad \text{(independence)}$$

$$= 1 - \left[\int_0^{\alpha - tn^{-\gamma}} \beta(1-\cancel{\beta}e^{-\beta\alpha})^{-1} e^{-\beta x}\,dx\right]^n$$

$$= 1 - \left[(1-e^{-\beta\alpha})^{-1}(1-e^{-\beta(\alpha-tn^{-\gamma})})\right]^n$$

$$= 1 - \left(\frac{1-e^{-\beta\alpha}e^{\beta tn^{-\gamma}}}{1-e^{-\beta\alpha}}\right)^n$$

$$= 1 - \left(1 + e^{-\beta\alpha}\,\frac{1-e^{\beta tn^{-\gamma}}}{1-e^{-\beta\alpha}}\right)^n$$

$$= 1 - \left(1 + \frac{e^{-\beta\alpha}}{1-e^{-\beta\alpha}}\left(-\beta tn^{-\gamma} + O(n^{-2\gamma})\right)\right)^n$$

$$= 1 - \left(1 - \frac{\beta t e^{-\beta\alpha}}{} + O(n^{-2\gamma})\right)^n$$

2007 Q5

$$\longrightarrow 1 - e^{-\left(\frac{\beta e^{-\beta\alpha}}{1-e^{-\beta\alpha}}\right)t} \quad \text{if we choose } \gamma = 1.$$

this differentiates to $\left(\frac{\beta e^{-\beta\alpha}}{1-e^{-\beta\alpha}}\right) \exp\left\{-\left(\frac{\beta e^{-\beta\alpha}}{1-e^{-\beta\alpha}}\right)t\right\}$

$$\therefore \quad n(\alpha - \hat{\alpha}) \xrightarrow{d} \text{Exp}\left(\frac{\beta e^{-\beta\alpha}}{1-e^{-\beta\alpha}}\right)$$

Secondly, to compute the asymptotic distn of $\hat{\beta}$, we

Taylor expand $\frac{\partial l}{\partial \beta}$ about β.

$$\left.\frac{\partial l}{\partial \beta}\right|_{\hat{\beta}} = \left.\frac{\partial l}{\partial \beta}\right|_{\beta} + (\hat{\beta}-\beta)\left.\frac{\partial^2 l}{\partial \beta^2}\right|_{\beta} + \frac{1}{2}(\hat{\beta}-\beta)^2 \left.\frac{\partial^3 l}{\partial \beta^3}\right|_{\xi_n}$$

ignoring the 2nd order term for now, we have that

$$\sqrt{n}(\hat{\beta}-\beta) = \frac{\sqrt{n}\left(\left.\frac{\partial l}{\partial \beta}\right|_{\beta} / n\right)}{-\left(\left.\frac{\partial^2 l}{\partial \beta^2}\right|_{\beta} / n\right)}$$

$$= \frac{\sqrt{n}\left(\frac{1}{\beta} - \frac{\hat{\alpha}e^{-\hat{\alpha}\beta}}{1-e^{-\hat{\alpha}\beta}} - \frac{\sum |X_i|}{n}\right)}{\frac{1}{\beta^2} - \frac{\hat{\alpha}^2 e^{-\beta\hat{\alpha}}}{(1-e^{-\beta\hat{\alpha}})^2}}$$

and note that

$$E|X_i| = 2\int^{\alpha} x\, 2^{-1}\beta(1-e^{-\beta\alpha})^{-1} e^{-\beta x}\, dx$$

$$= \cancel{\beta}(1-e^{-\beta\alpha})^{-1} \int_0^{\alpha} \beta e^{-\beta x}\, x \, dx$$

$$= (1-e^{-\beta\alpha})^{-1} \left\{ \left[-x e^{-\beta x} \right]_0^{\alpha} + \int_0^{\alpha} e^{-\beta x}\, dx \right\}$$

$$= (1-e^{-\beta\alpha})^{-1} \left\{ -\alpha e^{-\beta\alpha} + \frac{1}{\beta}(1-e^{-\beta\alpha}) \right\}$$

$$= \frac{1}{\beta} - \frac{\alpha e^{-\beta\alpha}}{(1-e^{-\beta\alpha})}$$

$\therefore$ Similarly,

$$E|X_i|^2 = 2\int_0^{\alpha} x^2\, 2^{-1} \beta (1-e^{-\beta\alpha})^{-1} e^{-\beta x}\, dx$$

$$= (1-e^{-\beta\alpha})^{-1} \left\{ \left[-x^2 e^{-\beta x} \right]_0^{\alpha} + 2\int_0^{\alpha} x e^{-\beta x}\, dx \right\}$$

$$= (1-e^{-\beta\alpha})^{-1} \left\{ -\alpha^2 e^{-\beta\alpha} + 2\left[-\frac{\alpha e^{-\beta\alpha}}{\beta} + \frac{1-e^{-\beta\alpha}}{\beta^2} \right] \right\}$$

$$= \frac{2(1-e^{-\beta\alpha}) - 2\beta\alpha e^{-\beta\alpha} - \beta^2\alpha^2 e^{-\beta\alpha}}{\beta^2(1-e^{-\beta\alpha})} = \frac{1}{\beta^2} - \frac{\alpha}{\beta}\frac{e^{-\beta\alpha}}{1-e^{-\beta\alpha}} - \alpha^2 \frac{e^{-\alpha\beta}}{1-e^{-\beta\alpha}}$$

$$\therefore \operatorname{Var}|X_i| = \frac{2(1-e^{-\beta\alpha}) - 2\beta\alpha e^{-\beta\alpha} - \beta^2\alpha^2 e^{-\beta\alpha}}{\beta^2(1-e^{-\beta\alpha})} - \frac{(1-e^{-\beta\alpha} - \alpha\beta e^{-\beta\alpha})^2}{\beta^2(1-e^{-\beta\alpha})^2}$$

$$= \frac{(2(1-e^{-\beta\alpha}) - 2\beta\alpha e^{-\beta\alpha} - \beta^2\alpha^2 e^{-\beta\alpha})(1-e^{-\beta\alpha}) - (1-e^{-\alpha\beta} - \alpha\beta e^{-\beta\alpha})^2}{\beta^2(1-e^{-\beta\alpha})^2}$$

$$= \frac{-(\cancel{\text{[illegible]}}\,\beta^2\alpha^2 e^{-\beta\alpha})(1-e^{-\beta\alpha}) + \cancel{2}(1-e^{-\beta\alpha})^2 \cancel{\text{[illegible]}} - \alpha^2\beta^2 e^{-2\beta\alpha}}{\beta^2(1-e^{-\beta x})^2}$$

$$= \frac{1}{\beta^2} - \alpha^2 \frac{e^{-\beta\alpha}}{(1-e^{-\beta\alpha})^2}$$

~~$= \frac{\alpha\beta(1-e^{-\alpha\beta})e^{-\alpha\beta} - \alpha^2\beta^2 e^{-\alpha\beta}\alpha\beta}{\beta^2(1-e^{-\beta\alpha})^2}$~~

~~$= \frac{\alpha\beta e^{-\alpha\beta} - \alpha\beta e^{-2\alpha\beta} - \alpha^2\beta^2 e^{-\alpha\beta}}{\beta^2(1-e^{-\beta\alpha})^2}$~~

~~$= \frac{\alpha\beta e^{-\alpha\beta}(1-e^{-\alpha\beta} - \alpha\beta)}{\beta^2(1-e^{-\beta\alpha})^2}$~~

Hence, by CLT,

$$\sqrt{n}\left(\frac{\sum |X_i|}{n} - \left(\frac{1}{\beta} - \frac{\alpha e^{-\beta\alpha}}{1-e^{-\beta\alpha}}\right)\right) \xrightarrow{d} N(0, \operatorname{Var}|X_i|).$$

But

$$\sqrt{n}(\hat\beta - \beta) = \frac{-\sqrt{n}\left(\frac{\sum|X_i|}{n} - \left(\frac{1}{\beta} - \frac{\alpha e^{-\beta\alpha}}{1-e^{-\beta\alpha}}\right)\right) + \sqrt{n}\left(\frac{1}{\beta} - \frac{\hat\alpha e^{-\hat\alpha\beta}}{1-e^{-\hat\alpha\beta}} - \left(\frac{1}{\beta} - \frac{\alpha e^{-\beta\alpha}}{1-e^{-\beta\alpha}}\right)\right)}{\frac{1}{\beta} + \frac{\hat\alpha^2 e^{-\beta\hat\alpha}}{(1-e^{-\beta\hat\alpha})^2}}$$

now note $\frac{1}{\beta^2} - \frac{\hat\alpha^2 e^{-\beta\hat\alpha}}{(1-e^{-\beta\hat\alpha})^2} \xrightarrow{p} \frac{1}{\beta^2} - \frac{\alpha e^{-\beta\alpha}}{(1-e^{-\beta\alpha})^2}$ by CMT, as $\hat\alpha$ is consistent.

$$\sqrt{n}\left(\frac{\alpha e^{-\beta\alpha}}{1-e^{-\beta\alpha}} - \frac{\hat\alpha e^{-\hat\alpha\beta}}{1-e^{-\hat\alpha\beta}}\right) =$$

$$= -\sqrt{n}(\hat\alpha - \alpha)\left(\frac{e^{-\hat\alpha\beta}}{1-e^{-\hat\alpha\beta}}\right) + \sqrt{n}\,\alpha\left(\frac{e^{-\beta\alpha}}{1-e^{-\beta\alpha}} - \frac{e^{-\beta\hat\alpha}}{1-e^{-\hat\alpha\beta}}\right)$$

$\xrightarrow{p} 0$ as $\sqrt{n}(\hat\alpha - \alpha) \xrightarrow{p} 0$ (since ~~&~~ $n(\hat\alpha - \alpha) = O_p$

and $\frac{e^{-\hat\alpha\beta}}{1-e^{-\hat\alpha\beta}} \xrightarrow{p} \frac{e^{-\alpha\beta}}{1-e^{-\alpha\beta}}$ by CMT.

Putting the pieces together and applying Slutsky's ,

$$\sqrt{n}\,(\hat{\beta}-\beta) \xrightarrow{d} \frac{N(0, Var(1X,1))}{\frac{1}{\beta^2} - \frac{\alpha^2 e^{-\beta\alpha}}{(1-e^{-\alpha\beta})^2}} ,$$

The va limiting variance is therefore

$$\cancel{\frac{Var(1X,1)}{\left(\frac{(1-e^{-\alpha\beta})^2 - \alpha^2\beta^2 e^{-\beta\alpha}}{\beta^2(1-e^{-\alpha\beta})}\right)^2}}$$

$$\sqrt{n}\,(\hat{\beta}-\beta) \xrightarrow{d} N\left(0, \frac{1}{\frac{1}{\beta^2} - \alpha^2\frac{e^{-\beta\alpha}}{1-e^{-\beta\alpha}}}\right)$$

proving part (iii)

2008 Q.6 Proper solution on next page.

Suppose we have n subjects. Then the likelihood is

$$L(\mu_1,\mu_2; X,Y) = \prod_{i=1}^{n} \frac{e^{-\mu_1}\mu_1^{x_i}}{x_i!} \cdot \frac{e^{-\mu_2}\mu_2^{y_i}}{y_i!}$$

$$\propto \exp\{(\Sigma X_i)\log\mu_1 + (\Sigma Y_i)\log\mu_2 - n\mu_1 - n\mu_2\}$$

~~let $H_0: \mu_1=\mu_2$, $H_1 = H_1: \mu$~~

let $H_0: \mu_1=\mu_2=\mu_0$ H_1: ~~$\frac{\mu_1}{\mu_2} = \gamma$ for where $\gamma \neq 1$ (fixed)~~

$H_1: \frac{\mu_1}{\mu_2} = e^{\gamma}$ where γ is fixed and $\neq 0$.

Let $\theta = \log\frac{\mu_1}{\mu_2}$, $\eta = \log\mu_2$, $U(X) = \Sigma X_i$, & $T(X) = \Sigma X_i + \Sigma Y_i$.

then $H_0: \theta=0$, $H_1: \theta=\gamma$ and

$$P_{\theta,\eta}(X,Y) \propto \exp\left\{(\Sigma X_i)\log\frac{\mu_1}{\mu_2} + (\Sigma X_i + \Sigma Y_i)\log\mu_2 + A(\theta,\eta)\right\}$$

$$= \exp\{U(X)\theta + T(X)\eta + A(\theta,\eta)\}.$$

Case 1: $\gamma > 0$. (i.e. $H_1: \mu_1 > \mu_2$)

By class results, $\exists$ a UMPU test of the form

$$\phi(U,T) = \begin{cases} 1 & \text{if } U > k(T) \\ \nu(T) & \text{if } U = k(T) \\ 0 & \text{o/w} \end{cases}$$

where $E_{\theta=0}\, \phi(U,T) \mid T = \alpha$ a.s.

~~i.e. $P\left(\Sigma X_i > k(\Sigma X_i + \Sigma Y_i) \mid \Sigma X_i + \Sigma Y_i\right) + E\left[\nu(T) 1_{\{u=k(T)\}} \mid T\right] = \alpha$~~

From the size constraint, ~~we obtain~~ and noting that

$$\cancel{E_{\mu_1=\mu_2}\left[\mathbb{1}_{\{\Sigma X_i >}\right.} \quad (\Sigma X_i \mid \Sigma X_i + \Sigma Y_i = t) \sim Bin(t, \tfrac{1}{2}),$$

we obtain $P(Bin(t, \frac{1}{2}) > k(t)) + \nu(t) P(Bin(t, \frac{1}{2}) = k(t)) = \alpha$,

$\therefore$ $k(t)$ is ~~Such~~ the unique integer s.t.

$$P(Bin(t, \tfrac{1}{2}) > k(t)) \leq \alpha \leq P(Bin(t, \tfrac{1}{2}) \geq k(t))$$

and $$\nu(t) = \frac{\alpha - P(Bin(t, \frac{1}{2}) > k(t))}{P(Bin(t, \frac{1}{2}) = k(t))}$$

This test can be formulated for any sample size n.

Now we impose that $P(\text{Type II error}) = \beta$.

$$E_{\theta=\gamma}\, \phi(U, T) = 1 - \beta,$$

$$\therefore E_{\theta=\gamma}\left[E_{\theta=\gamma}\left[\phi(U, T) \mid T \right]\right] = 1 - \beta$$

Note that under $\theta = \gamma$, $\frac{\mu_1}{\mu_2} = e^{\gamma}$ $\therefore$ $(\Sigma X_i \mid \Sigma X_i + \Sigma Y_i = t) \sim Bin(t, \frac{\mu_1}{\mu_1+\mu_2})$

$$= Bin(t, \frac{\mu_1}{\mu_1 + \mu_1 e^{-\gamma}}) = Bin(t, \frac{e^{\gamma}}{1+e^{\gamma}}),$$ and $\Sigma X_i + \Sigma Y_i \sim Poisson(\underbrace{n(\mu_1+\mu_2)}_{n\mu_2(1+e^{\gamma})})$

$$\cancel{\therefore E_{\theta=\gamma}\left[P(Bin(t, \tfrac{1}{2}) > k \right.}$$

let $f(t) = P(Bin(t, \frac{e^{\gamma}}{1+e^{\gamma}}) > k(t)) + \nu(t) P(Bin(t, \frac{e^{\gamma}}{1+e^{\gamma}}) = k(t))$.

2005 Q6

$X_1, \dots, X_n \sim$ Poisson λ $\quad H_0: \lambda = \lambda'$ $\quad$ • size

$Y_1, \dots, Y_n \sim$ Poisson λ' $\quad H_1: \lambda \neq \lambda'$ $\quad$ • power $\quad \Big\} \Longrightarrow n \geq ?$

• $d(\lambda, \lambda')$

~~$\bar{x} - \bar{y} \overset{d}{\approx} N(\lambda - \lambda', \frac{1}{n}\lambda(\lambda+\lambda')) \frac{1}{n}(\lambda+\lambda')$~~

~~$\bar{x} - \bar{y} \overset{d}{\approx} N(\lambda - \lambda'$~~

$$\bar{x} - \bar{y} \overset{d}{\approx} N\left(\lambda - \lambda', \frac{1}{n}(\lambda + \lambda')\right)$$

~~$E\, g(Y)$~~ Idea: use Δ-theorem to remove λ-dependence in variance

$E\, g(t) \approx g(\mu)$

$Var\, g(t) \approx \sigma^2 g'(\mu)^2$

Want $\quad 1 = \lambda g'(\lambda)^2 \Longrightarrow g(\lambda) = \sqrt{\lambda}$

so if $\quad Y \sim$ Poisson(λ), $\quad E\sqrt{Y} \approx \sqrt{\lambda}$, $\quad Var(\sqrt{Y}) \approx \frac{1}{4}$

so $Y \approx N(\sqrt{\lambda}, \frac{1}{4})$.

Thus $\quad \underbrace{\sqrt{\bar{x}} - \sqrt{\bar{y}}}_{T} \overset{d}{\approx} N\left(\sqrt{\lambda} - \sqrt{\lambda'}, \frac{1}{2n}\right)$

$\alpha = P_{\lambda = \lambda'}\left(|T| > \frac{1}{\sqrt{2n}} z_{1-\frac{\alpha}{2}}\right)$

$1 - \beta \leq P_{\lambda, \lambda'}\left(|T| > \frac{1}{\sqrt{2n}} z_{1-\frac{\alpha}{2}}\right) \quad \Longrightarrow 1 - \beta \leq P_{\lambda, \lambda'}\left(T - \delta > \frac{1}{\sqrt{2n}} z_{1-\frac{\alpha}{2}} - \delta\right)$

$\Longrightarrow 1 - \beta \leq P\left(N(0,1) > z_{1-\frac{\alpha}{2}} - \sqrt{2n}\,\delta\right)$

~~$\Longrightarrow 1 - \beta \leq P\left(\frac{1}{\sqrt{2n}} Z - \delta > \frac{1}{\sqrt{2n}} z_{1-\frac{\alpha}{2}}\right) \Longrightarrow 1 - \beta \leq \Phi\left(z_{1-\frac{\alpha}{2}} - \sqrt{2n}\,\delta\right)$~~

$Z \leftarrow N(0,1)$ $\qquad \delta \downarrow \sqrt{\lambda} - \sqrt{\lambda'}$

~~$\Longrightarrow \Phi(\dots) \leq \beta$~~

$\Longrightarrow z_{1-\beta} \leq -z_{1-\frac{\alpha}{2}} + \sqrt{2n}\,\delta$

$$\Longrightarrow \quad 2n \geq \left(\frac{z_{1-\frac{\alpha}{2}} + z_{1-\beta}}{\quad}\right)^2$$

2006 Q1

$$L(\lambda; Y) = \prod_{i=1}^{n} \frac{e^{-\lambda^{x_i}} \lambda^{x_i Y_i}}{Y_i!}$$

$$= e^{-\sum \lambda^{x_i}} \lambda^{\sum x_i Y_i} \prod_{i}^{n} \frac{1}{Y_i!}$$

$$= \exp\left\{ \left(\sum x_i Y_i\right) \log\lambda - \sum \lambda^{x_i} \right\} \prod \frac{1}{Y_i!}$$

$$\frac{L(\lambda; Y)}{L(\lambda; \tilde{Y})} = \lambda^{\sum x_i (Y_i - \tilde{Y}_i)} \cancel{\Pi} \frac{\prod \frac{1}{Y_i!}}{\prod \frac{1}{\tilde{Y}_i!}}$$

is independent of λ iff $\sum x_i Y_i = \sum x_i \tilde{Y}_i$

Also this is an exp fam with $\eta = \log\lambda$, $T(Y) = \sum x_i Y_i$, $B(\lambda) = \sum \lambda^{x_i}$

as $\tilde{\eta} = \{ \log\lambda : \lambda \in (0, \infty) \} = \mathbb{R}$ ~~is ep~~ has non-empty interior,

$T(X) = \sum x_i Y_i$ is C.S.

2006 Q2

$L(p_{01}, p_{10}; \vec{n}) \propto p_{01}^{n_{01}} (1-p_{01})^{n_{00}} p_{10}^{n_{10}} (1-p_{10})^{n_{11}}$

$\therefore \quad l(p_{01}, p_{10}) = n_{01} \log p_{01} + n_{00} \log(1-p_{01}) + n_{10} \log p_{10} + n_{11} \log(1-p_{10})$

$\therefore \quad \frac{\partial l}{\partial p_{01}} = \frac{n_{01}}{p_{01}} - \frac{n_{00}}{1-p_{01}} \quad \therefore \hat{p}_{01} = \frac{n_{01}}{n_{00}+n_{01}}$

$\frac{\partial^2 l}{\partial p_{01}^2} = -\frac{n_{01}}{p_{01}^2} - \frac{n_{00}}{(1-p_{01})^2} < 0$

$\frac{\partial l}{\partial p_{10}} = \frac{n_{10}}{p_{10}} - \frac{n_{11}}{1-p_{10}} \quad \therefore \hat{p}_{10} = \frac{n_{10}}{n_{10}+n_{11}}$

$\frac{\partial^2 l}{\partial p_{10}^2} = -\frac{n_{10}}{p_{10}^2} - \frac{n_{11}}{(1-p_{10})^2} < 0 \qquad \frac{\partial^2 l}{\partial p_{10} \partial p_{01}} = 0$

$\therefore \sup l(p_{01}, p_{10}) =$ ~~$n_{01} \log$~~

$= n_{01} \log \frac{n_{01}}{n_{00}+n_{01}} + n_{00} \log \frac{n_{00}}{n_{00}+n_{01}} + n_{10} \log \frac{n_{10}}{n_{11}+n_{10}} + n_{11} \log \frac{n_{11}}{n_{11}+n_{10}}$

Similarly $\sup_{p_{01}=p_{10}=p} l(p_{01}, p_{10}) = l\left(\frac{n_{01}+n_{10}}{n}, \frac{n_{01}+n_{10}}{n}\right)$

$= (n_{01}+n_{10}) \log \frac{n_{01}+n_{10}}{n} + (n_{11}+n_{00}) \log \frac{n_{11}+n_{00}}{n}$

~~& WA~~ By Wilks thm, $-2\log\Lambda \xrightarrow{d} \chi^2_1$ under $p_{01} = p_{10} = p$

(can prove this analogously to ~~[illegible]~~ appendix)

Now ~~consider~~ write ~~By~~ P_n for the measure under

See appendix for solution to a simplified version of this

problem when we consider a single RV
and its LRT under:

$X \sim Bin(n, p)$ $p = p_0$ vs $p = p_0 + \frac{\delta}{\sqrt{n}}$

2008 Q2 - APPENDIX 1

$$P_n(X) = \binom{n}{X} p_0^{X} (1-p_0)^{n-X}$$

$$Q_n(X) = \binom{n}{X} \left(p_0 + \frac{\delta}{\sqrt{n}}\right)^{X} \left(1-p_0-\frac{\delta}{\sqrt{n}}\right)^{n-X}$$

$$\log \frac{dQ_n}{dP_n} = X \log \frac{p_0+\frac{\delta}{\sqrt{n}}}{p_0} + (n-X)\log \frac{1-p_0-\frac{\delta}{\sqrt{n}}}{1-p_0}$$

$$= X \log\left(1+\frac{\delta}{p_0\sqrt{n}}\right) + (n-X)\log\left(1-\frac{\delta}{(1-p_0)\sqrt{n}}\right)$$

$$= X\left[\frac{\delta}{p_0\sqrt{n}} - \frac{\delta^2}{2p_0^2 n} + O(n^{-\frac{3}{2}})\right] + (n-X)\left[-\frac{\delta}{(1-p_0)\sqrt{n}} - \frac{\delta^2}{2(1-p_0)^2 n} + O(n^{-\frac{3}{2}})\right]$$

$$= \frac{\delta}{\sqrt{n}}\left(\frac{X}{p_0} - \frac{n-X}{1-p_0}\right) - \frac{\delta^2}{2n}\left(\frac{X}{p_0^2} + \frac{n-X}{(1-p_0)^2}\right) + o_p(1)$$

$$= \frac{\delta}{\sqrt{n}} \cdot \frac{X-np_0}{p_0(1-p_0)} - \frac{\delta^2}{2n}\cdot\left(\frac{X/n}{np_0^2} + \frac{1-X/n}{(1-p_0)^2}\right) + o_p(1)$$

$$= \frac{\delta}{p_0(1-p_0)} \underbrace{\sqrt{n}\left(\frac{X}{n}-p_0\right)}_{\xrightarrow{d} N(0,p_0(1-p_0))} - \frac{\delta^2}{2}\underbrace{\left(\frac{X/n}{p_0^2} + \frac{1-X/n}{(1-p_0)^2}\right)}_{\xrightarrow{p} \frac{1}{p_0}+\frac{1}{1-p_0} = \frac{1}{p_0(1-p_0)}} + o_p(1)$$

$\therefore$ LAN holds at p_0 (in particular, $P_n \triangleleft\triangleright Q_n$)

On the other hand,

$$-2\log\Lambda = 2X\log\frac{X}{n} + 2(n-X)\log\frac{n-X}{n} - 2X\log p_0 - 2(n-X)\log(1-p_0)$$

$$\xrightarrow{d} \chi^2_1 \text{ by Wilks thm}$$

But $-2\log\Lambda = 2X\log\frac{X}{np_0} + 2(n-X)\log\frac{n-X}{n(1-p_0)}$

$$= \left[\frac{2}{p_0(1-p_0)} - \left(\frac{X/n}{p_0^2} + \frac{1-X/n}{p(1-p_0)^2}\right)\right]\left[\sqrt{n}\left(\frac{X}{n} - p_0\right)\right]^2 + o_p(1)$$

(by calculation in Appendix 2)

As $\sqrt{n}\left(\frac{X}{n} - p_0\right) \xrightarrow{d} N(0, p_0(1-p_0))$, by bivariate Slutsky and CMT

we have that :

$$\left(-2\log\Lambda, \log\frac{dQ_n}{dP_n}\right) \xrightarrow[P_n]{d} \left(Z^2, -\frac{\delta^2}{2}\cdot\frac{1}{p_0(1-p_0)} + \frac{\delta}{\sqrt{p_0(1-p_0)}} Z\right), \quad Z \sim N(0,1)$$

$$\therefore \left(-2\log\Lambda, \frac{dQ_n}{dP_n}\right) \xrightarrow[P_n]{d} \left(Z^2, \exp\left\{-\frac{\delta^2}{2}\cdot\frac{1}{p_0(1-p_0)} + \frac{\delta}{\sqrt{p_0(1-p_0)}} Z\right\}\right)$$

By Le Cam's 3rd Lemma,

$$E_{Q_n} e^{it(-2\log\Lambda)} \longrightarrow E\left[e^{itZ^2} \cdot e^{-\frac{\delta^2}{2}\frac{1}{p_0(1-p_0)} + \frac{\delta}{\sqrt{p_0(1-p_0)}} Z}\right]$$

$$= e^{-\frac{\delta^2}{2}\frac{1}{p_0(1-p_0)}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{itZ^2 + \frac{\delta}{\sqrt{p_0(1-p_0)}} Z} e^{-\frac{Z^2}{2}} dZ$$

$$= e^{-\frac{\delta^2}{2}\frac{1}{p_0(1-p_0)}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[(1-2it)Z^2 - \frac{2\delta}{\sqrt{p_0(1-p_0)}} Z\right]\right\} dZ$$

$$= e^{--} \int \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2(1-2t)^{-1}}\left[Z - \frac{\delta}{(1-2t)\sqrt{p_0(1-p_0)}}\right]^2\right\} dZ \exp\left\{\frac{\delta^2}{2(1-2t)p_0(1-p_0)}\right\}$$

$$= \exp\left\{\frac{\delta^2}{2p_0(1-p_0)}\left(\frac{1}{1-2ti} - 1\right)\right\}\sqrt{(1-2ti)^{-1}}$$

$$= \frac{1}{\sqrt{1-2ti}} \exp\left\{\frac{\delta^2}{2p_0(1-p_0)}\,\frac{2ti}{1-2ti}\right\}$$

$$= (1-2ti)^{-\frac{1}{2}} \exp\left\{\frac{\delta^2}{p_0(1-p_0)} \cdot \frac{it}{1-2ti}\right\}$$

$\therefore -2\log\Lambda \xrightarrow[Q_n]{} $ non-central Chi-squared.

with 1 df and

2006 Q2 - APPENDIX 2 (WILKS thm for Binomial)

$$-\log\Lambda = X\log\frac{X}{n} + (n-X)\log\frac{n-X}{n} - X\log p_0 - (n-X)\log(1-p_0)$$

$$= X\log\frac{X}{np_0} + (n-X)\log\frac{n-X}{n(1-p_0)}$$

$$= X\left[\left(\frac{X}{np_0}-1\right) - \frac{\left(\frac{X}{np_0}-1\right)^2}{2} + O_p\left(\left(\frac{X}{np_0}-1\right)^3\right)\right]$$

$$+ (n-X)\left[\left(\frac{n-X}{n(1-p_0)}-1\right) - \frac{1}{2}\left(\frac{n-X}{n(1-p_0)}-1\right)^2 + O_p\left(\left(\frac{1-X}{n(1-p_0)}-1\right)^3\right)\right]$$

$$= \frac{X}{p_0}\left(\frac{X}{n}-p_0\right) + \frac{n-X}{1-p_0}\left(\frac{n-X}{n}-(1-p_0)\right)$$

$$- \frac{1}{2}X\frac{1}{p_0^2}\left(\frac{X}{n}-p_0\right)^2 - \frac{1}{2}(n-X)\frac{1}{(1-p_0)^2}\left(\frac{n-X}{n}-(1-p_0)\right)^2$$

$$+ X\,O_p\left(\left(\frac{X}{n}-p_0\right)^3\right) + (n-X)O_p\left(\left(\frac{n-X}{n}-(1-p_0)\right)^3\right)$$

$$= \left(\frac{X}{p_0} - \frac{n-X}{1-p_0}\right)\left(\frac{X}{n}-p_0\right) - \frac{1}{2}\left(\frac{X}{p_0^2} + \frac{n-X}{(1-p_0)^2}\right)\left(\frac{X}{n}-p_0\right)^2$$

$$+ \frac{1}{\sqrt{n}}\frac{X}{n}O_p\left(\left[\sqrt{n}\left(\frac{X}{n}-p_0\right)\right]^3\right) + \frac{1}{\sqrt{n}}\left(\frac{n-X}{n}\right)O_p(1)$$

$$= \left(\frac{X-p_0n}{p_0(1-p_0)}\right)\left(\frac{X}{n}-p_0\right) - \frac{1}{2}\left(\frac{X-2p_0X+p_0^2X+p_0n-p_0X}{p_0^2(1-p_0)^2}\right)\left(\frac{X}{n}-p_0\right)^2$$

$$+ \frac{1}{\sqrt{n}}O_p(1) \quad \text{(Slutsky's)}$$

$$= \frac{1}{p_0(1-p_0)}\left(\frac{X}{n}-p_0\right)^2\cdot n - \frac{1}{2}\left(\frac{X/n}{p_0^2} + \frac{1-X/n}{(1-p_0)^2}\right)\left[\sqrt{n}\left(\frac{X}{n}-p_0\right)\right]^2 + o_p(1)$$

$$\xrightarrow{d} \frac{1}{p_0(1-p_0)}N(0,p_0(1-p_0))^2 - \frac{1}{2}\left(\frac{1}{p_0}+\frac{1}{1-p_0}\right)N(0,p_0(1-p_0))^2$$

$$= \frac{1}{2}N(0,1)^2 \quad \therefore \quad \Lambda \to \chi^2_1 \quad \square$$

$$\nearrow N(0, p_0(1-p_0))$$

$$P_{p_0}\left(\sqrt{n}\left(\frac{X}{n} - p_0\right) < t\right) \to \Phi\left(t/\sqrt{p_0(1-p_0)}\right)$$

$$P_{p_0+\frac{\delta}{\sqrt{n}}}\left(\sqrt{n}\left(\frac{X}{n} - p_0\right) \le t\right) = P_{p_0+\frac{\delta}{\sqrt{n}}}\left(\sqrt{n}\left(\frac{X}{n} - \left(p_0 + \frac{\delta}{\sqrt{n}}\right)\right) \le t - \delta\right)$$

$$\to \Phi\left(\frac{t-\delta}{\sqrt{p_0(1-p_0)}}\right)$$

$$\therefore \quad \sqrt{n}\left(\frac{X}{n} - p_0\right) \xrightarrow[p_0+\frac{\delta}{\sqrt{n}}]{} N\left(\delta, \sqrt{p_0(1-p_0)}\right)$$

— see exercise 6.33 TPE

2006 Q4

(a) By assumption, in polar coordinates we have

$$Y_i = (R_i, \theta_i) \quad \text{where} \quad \left.\begin{array}{l} R_i \sim f_R(\cdot) \\ \theta_i \sim U(0, 2\pi) \end{array}\right\} \text{independently}$$

$$\|Y_1 - Y_2\| = \|(R_1\cos\theta_1, R_1\sin\theta_1) - (R_2\cos\theta_2, R_2\sin\theta_2)\|$$

$$= \sqrt{(R_1\cos\theta_1 - R_2\cos\theta_2)^2 + (R_1\sin\theta_1 - R_2\sin\theta_2)^2}$$

$$= \sqrt{R_1^2\cos^2\theta_1 - 2R_1R_2\cos\theta_1\cos\theta_2 + R_2^2\cos^2\theta_2 + R_1^2\sin^2\theta_1 - 2R_1R_2\sin\theta_1\sin\theta_2 + R_2^2\sin^2\theta_2}$$

$$= \sqrt{\left(R_1^2 + R_2^2 - 2R_1R_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)\right)}$$

~~Claim: $\delta(\vec{Y}) = \frac{1}{\binom{n}{2}}\sum_{i \neq j} \|Y_{(i)} - Y_{(j)}\|$ is the UMVUE.~~

~~$Y_{(i)} = (\theta_i, R$~~ $Y_{(i)}$ is the point corresponding to the ith order statistic in R_i.

$$p_\theta(\vec{y}) = \prod_{i=1}^{n} \frac{1}{2\pi} f(r_i) \qquad \theta_i \in [0, 2\pi), \quad r_i \in (0, \infty)$$

By class results, $R_{(1)}, \ldots, R_{(n)}$ is C.S. * ???

$$\|Y_1 - Y_2\|^2 = R_1^2 + R_2^2 - 2R_1R_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$

$$\therefore \; E\|Y_1 - Y_2\|^2 = 2ER_1^2 - 2E^2R_1\,E(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$

alternatively, $E\|Y_1 - Y_2\|^2 = \underbrace{E\|Y_1\|^2 + E\|Y_2\|^2}_{2E\|Y_1\|^2} + \underbrace{E\langle Y_1, Y_2\rangle}_{=0 \text{ since } Y_1 \overset{d}{=} -Y_1}$

Now note that

~~$E \cos\theta_1 \sin$~~

$$E \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 = 2 E^2 \cos\theta_1 + E^2 \sin\theta_1$$

$$= \left(\int_0^{2\pi} \cos\theta_1 \, d\theta\right)^2 + \left(\int_0^{2\pi} \sin\theta_1 \, d\theta_1\right)^2$$

$$= 0$$

Hence $E \|Y_1 - Y_2\|^2 = 2 E R_1^2$

Thus $\delta(\vec{Y}) = 2 \frac{1}{n} \sum_{i=1}^{n} R_{(i)}^2$ is UMVUE.

(unbiased func of C.S. statistic)

Aside: To show that $(R_{(1)}, \dots, R_{(n)})$ is C.S.

Suppose $E_{R \sim f} h(R_{(1)}, \dots, R_{(n)}) = 0 \quad \forall f$

then pick ~~f_θ~~ $f(\vec{r}) = C(\theta_1, \dots, \theta_n) e^{-\theta_1 \sum_1^n r_i - \theta_2 \sum_1^n r_i^2 - \dots - \theta_n \sum_1^n r_i^n - \sum_1^n r_i^{2n}}$

for any $\vec{\theta} \in \mathbb{R}^n$. By exp. fam. theory, ~~$(\sum x_i, \sum x_i^2, \dots, \sum x_i^n)$ is C.S.~~

$T = (\sum r_i, \sum r_i^2, \dots, \sum r_i^n)$ is C.S.

By standard argument, T is a bijection of $(R_{(1)}, \dots, R_{(n)})$

(Suppose $\sum r_i = \sum s_i, \sum r_i^2 = \sum s_i^2, \dots, \sum r_i^n = \sum s_i^n \}$, then ...

By some convoluted argument $\Longrightarrow (r_1, \dots, r_n) = (s_1, \dots, s_n)$)

2008 Q5

(a) The likelihood is

$$L(\theta; n_1, n_2, n_3) = (\theta^2)^{n_1} (2\theta(1-\theta))^{n_2} ((1-\theta)^2)^{n_3} \binom{n}{n_1, n_2, n_3}$$

$$= 2^{n_2} \theta^{2n_1+n_2} (1-\theta)^{n_2+2n_3} \binom{n}{n_1, n_2, n_3}$$

$$= \exp\{(2n_1+n_2)\log\theta + (n_2+2n_3)\log(1-\theta)\} 2^{n_2} \binom{n}{n_1, n_2, n_3}$$

$$= \exp\{(n+(n_1-n_3))\log\theta + (n-(n_1-n_3))\log(1-\theta)\} 2^{n_2} \binom{n}{-}$$

Ⓘ $$= \exp\{(n_1-n_3)\log\frac{\theta}{1-\theta} + n\log\theta(1-\theta)\} 2^{n_2} \binom{n}{-}$$

this is a 1 parameter exp. fam. with

$T(n_1, n_2, n_3) = n_1 - n_2$, $\eta(\theta) = \log\frac{\theta}{1-\theta}$, $B(\theta) = n\log\theta(1-\theta)$

as $\theta \in (0,1)$, $\log\frac{\theta}{1-\theta} \in \mathbb{R}$ with non-empty interior.

$\therefore$ $T = n_1 - n_3$ is MS.

(b) Marginally, $n_1 \sim Bin(n, \theta^2)$ $n_3 \sim Bin(n, (1-\theta)^2)$

$\therefore$ $ET = n\theta^2 - n(1-\theta)^2 = -n + 2n\theta$

$\therefore$ $E\frac{T+n}{2n} = \theta$

$\therefore$ $\frac{T+n}{2n}$ is UMVUE.

$$= \frac{n_1 - n_3 + n}{2n} = \frac{2n_1 + n_2}{2n}$$

(c) if $\theta \in (0,1)$, from I,

$$\ell(\theta; n_1, n_2, n_3) = (n_1 - n_3)\log\frac{\theta}{1-\theta} + n\log(\theta(1-\theta))$$

$$\therefore \quad \frac{\partial \ell}{\partial \theta} = \frac{n_1 - n_3}{\theta} + \frac{n_1 - n_3}{1-\theta} + \frac{n}{\theta} - \frac{n}{1-\theta}$$

$$\therefore \quad \frac{\partial^2 \ell}{\partial \theta^2} = -\frac{n_1 - n_3}{\theta^2} + \frac{n_1 - n_3}{(1-\theta)^2} - \frac{n}{\theta^2} - \frac{n}{(1-\theta)^2}$$

$$= -\frac{n + n_1 - n_3}{\theta^2} - \frac{n + n_3 - n_1}{(1-\theta)^2} < 0 \quad \forall \theta \in (0,1)$$

$\therefore$ unique maximiser is at stationary point :

$$\frac{\partial \ell}{\partial \theta} = 0 \implies \frac{n_1 - n_3 - n}{1-\theta} + \frac{n + n_1 - n_3}{\theta} = 0$$

$$\implies \theta(n_1 - n_3 - n) - \theta(n + n_1 - n_3) = -(n + n_1 - n_3)$$

$$\implies \theta(-2n) = -n - n_1 + n_3$$

$$\implies \hat\theta = \frac{n + n_1 - n_3}{2n} = \frac{2n_1 + n_2}{2n} \quad (= \text{UMVUE})$$

$$\hat\theta = \frac{2\sum_{i=1}^{n} \mathbb{1}_{\{X_i = A/A\}} + \sum_{i=1}^{n} \mathbb{1}_{\{X_i = A/B\}}}{2n}$$

$$= \frac{\sum_{i=1}^{n} \mathbb{1}_{\{X_i = A/A\}} + \mathbb{1}_{\{X_i = A/B\}}/2}{n}$$

$$= \frac{\sum_{i=1}^{n} Y_i}{n} \qquad \text{where} \quad Y_i = \mathbb{1}_{\{X_i = A/A\}} + \frac{1}{2}\mathbb{1}_{\{X_i = A/B\}}$$

$$E\,Y_i = \theta^2 + \theta(1-\theta) = \theta$$

$$E\,Y_i^2 = E\,\mathbb{1}_{\{X_i = A/A\}}^2 + E\,\mathbb{1}_{\{X_i = A/A\}}\mathbb{1}_{\{X_i = A/B\}} + \frac{1}{4}E\,\mathbb{1}_{\{X_i = A/B\}}^2$$

$$= \theta^2 + \frac{1}{4}(2\theta(1-\theta)) = \theta^2 + \frac{1}{2}\theta(1-\theta) = \frac{1}{2}\theta^2 + \frac{1}{2}\theta = \frac{1}{2}\theta(\theta+1)$$

~~$\therefore$ By CLT, $\sqrt{n}(\hat\theta - \theta) \xrightarrow{d} N(0, \frac{1}{2}\theta(\theta+1))$~~

2006 Q5

$\therefore$ Var Y_i = $\frac{1}{2}\theta^2 + \frac{1}{2}\theta - \theta^2 = \frac{1}{2}\theta - \frac{1}{2}\theta^2 = \frac{1}{2}\theta(1-\theta)$

$\therefore$ By CLT

$$\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, \tfrac{1}{2}\theta(1-\theta))$$

(if $\theta = 0$, $\hat{\theta} \equiv 0$, if $\theta = 1$, $\hat{\theta} \equiv 1$ w.p. 1)

(d) Now our likelihood is

$$L(\theta; n_2, n_4) = \binom{n}{n_2} (2\theta(1-\theta))^{n_2} (\theta^2 + (1-\theta)^2)^{n_4}$$

$\therefore$ $n_2 \sim Bin(n, 2\theta(1-\theta))$

Let $p = 2\theta(1-\theta) = 2\theta - 2\theta^2 = -2(\theta-\frac{1}{2})^2 + \frac{1}{2}$

as $\theta \in [0, \frac{1}{2}]$, $\theta \mapsto p(\theta)$ is bijective and

$$\theta = -\sqrt{\frac{\frac{1}{2}-p}{2}} + \frac{1}{2}$$

By class results, $\hat{p}_{MLE} = \min(\frac{n_2}{n}, \frac{1}{2})$

By invariance of MLE, $\hat{\theta}_{MLE} = -\sqrt{\frac{\frac{1}{2} - \min(\frac{n_2}{n}, \frac{1}{2})}{2}} + \frac{1}{2}$

for $p \in [0, \frac{1}{2}]$, $\sqrt{n}(\frac{n_2}{n} - p) \to N(0, p(1-p))$ by CLT.

$\therefore$ ~~for $p \in [0, \frac{1}{2})$~~ $\therefore$ for $p \in [0, \frac{1}{2})$, $\hat{p}_{MLE} = \frac{n_2}{n}$ w.h.p and so, $\sqrt{n}(\hat{p}_{MLE}-p) \longrightarrow N(0, p(1-p))$

Let $g(x) = \frac{1}{2} - \sqrt{\frac{\frac{1}{2}-x}{2}}$ so $g'(x) = \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{\frac{1}{2}-x}{2}}} \cdot \left(+\frac{1}{2}\right) = \frac{1}{4\sqrt{\frac{1/2-x}{2}}}$

By Δ-method, for $\theta \in (0, \frac{1}{2})$,

$$\sqrt{n}(\hat{\theta}_{MLE}-\theta) \xrightarrow{d} N\left(0, p(1-p) \cdot \frac{1}{16\sqrt{\frac{1/2-p}{2}}^{2}}\right) = N\left(0, \frac{p(1-p)}{4(1-2p)}\right)$$

and $\frac{p(1-p)}{4(1-2p)} = \frac{2\theta(1-\theta)(1-2\theta(1-\theta))}{4(1-4\theta(1-\theta))} =$

For $p = \frac{1}{2}$, as $\frac{n_2}{n} - \frac{1}{2} \xrightarrow{d} N(0, 1/4)$,

$$\hat{p}_{MLE} = \begin{cases} \frac{n_2}{n} & \text{w.p. } 1/2 \\ 1/2 & \text{w.p. } 1/2 \end{cases}$$

$$\therefore \hat{\theta}_{MLE} = \frac{1}{2} - \sqrt{\frac{\max(0, \frac{1}{2} - \frac{n_2}{n})}{2}} = \frac{1}{2} - \sqrt{\frac{\frac{1}{2} - \hat{p}_{MLE}}{2}}$$

$$\sqrt{n}\left(\frac{1}{2} - \frac{n_2}{n}\right) \longrightarrow N(0, 1/4)$$

$$\therefore n^{1/4}\left(\hat{\theta}_{MLE} - \frac{1}{2}\right) = -n^{1/4}\sqrt{\frac{\max(0, \frac{1}{2} - \frac{n_2}{n})}{2}} = -\sqrt{\frac{\max(0, \sqrt{n}(\frac{1}{2} - \frac{n_2}{n}))}{2}}$$

$$\xrightarrow{d} -\sqrt{\frac{\max(0, N(0, 1/4))}{2}}$$

$$\therefore \underbrace{n^{1/4}\left(\hat{\theta}_{MLE} - \frac{1}{2}\right) \xrightarrow{d}}_{} \begin{cases} 0 & \text{w.p. } 1/2 \\ -\frac{1}{2}\sqrt{N(0,1)} & \text{w.p. } 1/2, \ \big| N(0,1) > 0 \end{cases}$$

$$\xrightarrow{d} \; = -\frac{1}{2}\sqrt{N(0,1)_+}$$

2006 Q6

(a) $L(\mu; x) \propto \exp\{-\frac{1}{2}\Sigma(x_i-\mu)^2]$

$$l(\mu; x) = -\frac{1}{2}\Sigma(x_i-\mu)^2 = -\frac{n}{2}(\bar{x}-\mu)^2 + \text{constant}$$

to minimize the quadratic, we pick μ as close as possible to $\bar{x}$. $\therefore \hat{\mu}_{MLE} = \bar{\mu}$ (nearest integer to $\bar{x}$)

(b) $E\,\bar{\mu} = \sum_{z\in\mathbb{Z}} z\,P(\bar{x} \in (z-\frac{1}{2}, z+\frac{1}{2}))$.

$$\therefore\ E[\bar{\mu}-\mu] = \sum_{z\in\mathbb{Z}} (z-\mu)\,P(\bar{x} \in (z-\tfrac{1}{2}, z+\tfrac{1}{2}))$$

$$= \sum_{z\in\mathbb{Z}} (z-\mu)\,P(\bar{x}-\mu \in ((z-\mu)-\tfrac{1}{2}, (z-\mu)+\tfrac{1}{2}))$$

$$= \sum_{z\in\mathbb{Z}} z\,P(N(0,\tfrac{1}{n}) \in (z-\tfrac{1}{2}, z+\tfrac{1}{2}))$$

$$= \sum_{z\in\mathbb{Z}^+} z\,P(N(0,\tfrac{1}{n}) \in (z-\tfrac{1}{2}, z+\tfrac{1}{2})) + \sum_{z\in\mathbb{Z}^-} z\,P(N(0,\tfrac{1}{n}) \in (z-\tfrac{1}{2}, z+\tfrac{1}{2}))$$

$$\overset{\textcircled{\scriptsize II}}{=} \sum_{z\in\mathbb{Z}^+} z\left[P(N(0,\tfrac{1}{n}) \in (z-\tfrac{1}{2}, z+\tfrac{1}{2})) - P(+N(0,\tfrac{1}{n}) \in (z-\tfrac{1}{2}, z+\tfrac{1}{2}))\right]$$

$$= 0 \qquad \square$$

~~$\sqrt{n}(\bar{\mu}-\mu)$~~

$$P(\sqrt{n}(\bar{\mu}-\mu) \le t) = P(\bar{\mu} \le \mu + \tfrac{t}{\sqrt{n}})$$

$$P(|\sqrt{n}(\bar{\mu}-\mu)| \le t) = P(\bar{\mu} \in (\mu - \tfrac{t}{\sqrt{n}}, \mu + \tfrac{t}{\sqrt{n}})) \geq P(\bar{x} \in (\mu-\tfrac{1}{2}, \mu+\tfrac{1}{2}))$$

$$\longrightarrow 1 \text{ by WLLN}$$

(c) $E\,\overline{[x]} = \frac{1}{n}\sum E[X_i] = E([X]) =$

$$= \sum_{z\in\mathbb{Z}} z\,P\left(X_1 \in (z-\tfrac{1}{2}, z+\tfrac{1}{2})\right)$$

$$\therefore\ E([X_i]-\mu) = \sum_{z\in\mathbb{Z}} (z-\mu)\,P\left(X_1 \in (z-\tfrac{1}{2}, z+\tfrac{1}{2})\right)$$

$$= \sum_{z\in\mathbb{Z}} (z-\mu)\,P\left(X_i-\mu \in (z-\mu-\tfrac{1}{2}, z-\mu+\tfrac{1}{2})\right)$$

$$\overset{\textcircled{\scriptsize I}}{=} \sum_{z\in\mathbb{Z}} z\,P\left(N(0,1) \in (z-\tfrac{1}{2}, z+\tfrac{1}{2})\right)$$

$= 0$ by the same calculation in (b).

By WLLN, $\overline{[x]} \xrightarrow{P} \mu$.

(d) No longer holds as I does not go through if $\mu \notin \mathbb{Z}$.

$\therefore$ the cancellation from II does not happen, and so

$[X_i]$ is biased $\therefore$ $\overline{[x]}$ ~~is biased~~ has constant

non-zero bias $\therefore$ it is not consistent.

2005 Q1

(a) $P(Y_i = 1) = \theta^2 + (1-\theta)^2 = 1 - 2\theta + 2\theta^2 = 2(\theta - \frac{1}{2})^2 + \frac{1}{2}$

$P(Y_i = -1) = 2\theta(1-\theta) = 2\theta - 2\theta^2 = \frac{1}{2} - 2(\theta - \frac{1}{2})^2$

So $Y_i \overset{iid}{\sim} 2B(1, 1-2\theta+2\theta^2) - 1$

$$p_\theta(\vec{y}) = \prod_{i=1}^{n} (1-2\theta+2\theta^2)^{\mathbb{1}\{y_i=1\}} (2\theta - 2\theta^2)^{\mathbb{1}\{y_i=-1\}}$$

$$= (1-2\theta+2\theta^2)^{\sum \mathbb{1}\{y_i=1\}} (2\theta-2\theta^2)^{\sum \mathbb{1}\{y_i=-1\}}$$

$$= (1-2\theta+2\theta^2)^{\sum \mathbb{1}\{y_i=1\}} (2\theta-2\theta^2)^{n - \sum \mathbb{1}\{y_i=1\}}$$

$$= \exp\left\{ \log\left(\frac{1-2\theta+2\theta^2}{2\theta-2\theta^2}\right) \sum_1^n \mathbb{1}\{y_i=1\} + n\log(2\theta-2\theta^2) \right\}$$

this is an exp fam with natural parameter

$\eta(\theta) = \log\left(\frac{1-2\theta+2\theta^2}{2\theta-2\theta^2}\right)$ as $\theta \in (0,1)$, ~~$\eta(\theta) = \mathbb{R}$~~

$\{\eta(\theta) : \theta \in (0,1)\} = \mathbb{R}$ which has non-empty interior.

(By class results, $T(\vec{Y}) = \sum_1^n \mathbb{1}\{Y_i=1\}$ is M.S. and C.S.

(note this is just a binomial w success p. $1-2\theta+2\theta^2$).

(b) Note that ~~$\psi(\theta)$~~ $E\,T = \sum E\,\mathbb{1}_{\{Y_i=1\}} = n(1-2\theta-2\theta^2)$
$= n(1-2\theta(1-\theta)) = n(1-2\psi)$.

$\therefore \quad \frac{1}{2}\left(1-\frac{T}{n}\right)$ is unbiased fun of the C.S statistic

$\therefore \quad \delta(Y) = \frac{1}{2}\left(1-\frac{\sum \mathbb{1}\{Y_i=1\}}{n}\right)$ is UMVUE.

$\mathrm{Var}\ \delta(Y) = \frac{1}{4}\mathrm{Var}\ \frac{\sum \mathbb{1}\{Y_i=1\}}{n} = \frac{1}{4n^2}\mathrm{Var}\ \overbrace{\sum \mathbb{1}_{\{Y_i=1\}}}^{B(n,1-2\theta-2\theta^2)} = \frac{1}{4n^2}\cdot n(1-2\psi)(1\psi)$

$= \frac{1}{n}\left(\frac{1}{2}-\psi\right)\psi$

(c) Let $\rho = 2(\theta-\frac{1}{2})^2+\frac{1}{2}$. In essence, we have a Binomial for n,ρ and our MLE for ρ is $\frac{\sum \mathbb{1}\{Y_i=1\}}{n}$

(likelihood is concave in ρ)

Once we restrict to $\rho \in (\frac{1}{2},1)$, then our likelihood is still concave, so MLE for ρ is

$$\max\left\{\frac{1}{2}, \frac{\sum \mathbb{1}\{Y_i=1\}}{n}\right\}$$ (if stationary point is at $\rho < \frac{1}{2}$, choose boundary).

Our current problem with $\theta \in (0,\frac{1}{2})$ is a ~~re~~ 1-1 reparametrization

~~$\rho(\theta)$~~ $\theta(\rho) = -\sqrt{\frac{\rho-\frac{1}{2}}{2}}+\frac{1}{2}$ for $\rho \in (\frac{1}{2},1) \rightarrow \theta \in (0,\frac{1}{2})$

By class results,

$$\hat{\theta}_{MLE} = -\sqrt{\max\left\{\frac{1}{2}, \frac{\sum \mathbb{1}\{Y_i=1\}}{n}\right\}-\frac{1}{2}}+\frac{1}{2}$$

2005 Q.1

$$\therefore \quad 2\left(\hat{\theta}_{MLE} - \frac{1}{2}\right)^2 = \max\left\{\frac{1}{2}, \frac{\sum \mathbb{1}\{Y_i=1\}}{n}\right\} - \frac{1}{2}$$

$$\therefore \quad 2\left(\hat{\theta}_{MLE} - \frac{1}{2}\right)^2 + \frac{1}{2} = \max\left\{\frac{1}{2}, \frac{\sum \mathbb{1}\{Y_i=1\}}{n}\right\}$$

Now split into cases.

Case 1: ~~$\theta \in (\frac{1}{2}, 1)$~~ $\theta \in (0, \frac{1}{2})$. Then

$$\sqrt{n}\left(\frac{\sum \mathbb{1}\{Y_i=1\}}{n} - p\right) \xrightarrow{d} N(0, p(1-p))$$

i.e.

$$\sqrt{n}\left(\frac{\sum \mathbb{1}\{Y_i=1\}}{n} - \frac{1}{2} - 2\left(\theta - \frac{1}{2}\right)^2\right) \xrightarrow{d} N\left(0, (1-2\theta+2\theta^2)(2\theta-2\theta^2)\right)$$

$\therefore$ w.h.p. $\frac{\sum \mathbb{1}\{Y_i=1\}}{n} > \frac{1}{2}$ and so

$$2\left(\hat{\theta}_{MLE} - \frac{1}{2}\right)^2 + \frac{1}{2} = \frac{\sum \mathbb{1}\{Y_i=1\}}{n}$$

Ⓘ and so $\sqrt{n}\left[2\left(\hat{\theta}_{MLE} - \frac{1}{2}\right)^2 - 2\left(\theta - \frac{1}{2}\right)^2\right] \xrightarrow{d} N\left(0, (1-2\theta+2\theta^2)(2\theta-2\theta^2)\right)$

Alternatively, can write $g(p) = \frac{1}{2} - \sqrt{\frac{p-\frac{1}{2}}{2}}$ so that

$g'(p) = -\frac{1}{2}\sqrt{\frac{2}{p-\frac{1}{2}}} \cdot \frac{1}{2} = -\frac{1}{2\sqrt{2p-1}}$ and so, for $p \in (\frac{1}{2}, 1)$,

By Δ-method, if ~~$\theta \in (\frac{1}{2}, 1)$~~ $\theta \in (0, \frac{1}{2})$

$$\sqrt{n}\left(\hat{\theta}_{MLE} - \theta\right) \xrightarrow{d} N\left(0, \frac{p(1-p)}{4(2p-1)}\right) = N\left(0, \frac{(1-2\theta+2\theta^2)(2\theta-2\theta^2)}{4(1-4\theta+4\theta^2)}\right)$$

Case 2: If $\theta = 0$, then $\hat{\theta} = 0$ a.s.

Case 3: If $\theta = \frac{1}{2}$, then $\hat{\theta} = \frac{1}{2}$ w.p. $\frac{1}{2}$

as $\frac{\sum \mathbb{1}\{Y_i = 1\}}{n} < \frac{1}{2}$ w.p. $\frac{1}{2}$

Hence, by I,

$$\sqrt{n}\left(2(\hat{\theta}_{MLE} - \tfrac{1}{2})^2\right) \xrightarrow{d} \left[N(0, \tfrac{1}{4})\right]_+$$

$$\left(\text{as } \sqrt{n}\left(\max\left\{\tfrac{1}{2}, \tfrac{\sum \mathbb{1}\{Y_i = 1\}}{n}\right\} - \tfrac{1}{2}\right) \xrightarrow{d} \left[N(0, \tfrac{1}{4})\right]_+ \right)$$

$$\therefore \quad n^{\frac{1}{4}}(\hat{\theta}_{MLE} - \tfrac{1}{2})\sqrt{2} \xrightarrow{d} -\sqrt{\left[N(0, \tfrac{1}{4})\right]_+}$$

$$n^{\frac{1}{4}}(\hat{\theta}_{MLE} - \tfrac{1}{2}) \xrightarrow{d} -\frac{1}{\sqrt{2}}\sqrt{\tfrac{1}{2} N(0,1)_+} = -\frac{1}{2}\sqrt{N(0,1)_+}$$

2005 Q2

(a) $P_{\theta=0}(\phi=1) = \alpha$

$\Longrightarrow P_{\theta=0}(Y_1 \geq k) = \alpha$

$\Longrightarrow (1-k)^n = \alpha \Rightarrow \boxed{k = 1-\alpha^{1/n}}$

(b) $P_\theta(\phi=1) = 1$ if $\theta \geq 1$, as $Y_n \geq 1$ w.p. 1

If ~~$\theta <$~~ $\theta < 1$, $\theta > k$, $\overbrace{P_\theta(\phi=1)}^{\text{as } Y_1 \geq k \text{ w.p. 1.}} = 1$. If $\theta < k$,

$P_\theta(\phi=1) = P_\theta(Y_1 \geq k \text{ or } Y_n \geq 1)$

$= P_\theta(Y_n \geq 1) + P(Y_1 \geq k \text{ and } Y_n < 1)$

$= 1 - P_\theta(Y_n \leq 1) + P_\theta(Y_i \in (k,1)\ \forall i)$

$= 1 - (1-\theta)^n + (1-k)^n$

$= 1 - (1-\theta)^n + \alpha$

$$\therefore P_\theta(\phi=1) = \beta_\theta = \begin{cases} 1-(1-\theta)^n + \alpha & \text{if } \theta \in [0, 1-\alpha^{1/n}] \\ 1 & \text{if } \theta > 1-\alpha^{1/n}. \end{cases}$$

(c) Fix the alternative $\theta = \theta_1 > 0$.

If $\theta_1 \geq 1-\alpha^{1/n}$, ϕ has power 1 so is MP.

Otherwise, if $\theta_1 \in (0, 1-\alpha^{1/n})$, then

$$\frac{p_{\theta_1}(X)}{p_{\theta=0}(X)} = \frac{\mathbb{1}\{Y_1 \geq \theta_1\}\,\mathbb{1}\{Y_n \leq \theta_1 + 1\}}{\mathbb{1}\{Y_1 \geq 0\}\,\mathbb{1}\{Y_n \leq 1\}}$$

Thus, choosing $k=1$ in NP lemma,

$$\frac{p_{\theta_1}(X)}{p_0(X)} > 1 \Rightarrow Y_n > 1 \Rightarrow \phi = 1$$

& $$\frac{p_{\theta_1}(X)}{p_0(X)} < 1 \Rightarrow Y_1 < \theta_1 \Rightarrow Y_1 < k \Rightarrow \phi = 0.$$

$\therefore$ ϕ is of NP form $\forall \theta_1$ $\therefore$ ϕ is UMP $\square$

(d) ? Any values will do.

2005 Q4

$p(\theta_1, \ldots, \theta_n, \sigma, \tau | y) \propto p(y|\theta)\,\pi(\theta_j|\tau)\,p(\sigma,\tau)$

$$\propto \left\{\prod_{i=1}^{2}\prod_{j=1}^{J}(2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}(y_{ij}-\theta_j)^2}\right\}\left\{\prod_{j=1}^{J}(2\pi\tau^2)^{-\frac{1}{2}} e^{-\frac{1}{2\tau^2}\theta_j^2}\right\}$$

$$\propto \sigma^{-2J}|\tau|^{-J}\exp\left\{\sum_{ij}-\frac{1}{2\sigma^2}(y_{ij}-\theta_j)^2+\sum_{j}-\frac{1}{2\tau^2}\theta_j^2\right\}$$

$$\propto \sigma^{-2J}|\tau|^{-J}\exp\left\{-\frac{\sum y_{ij}^2}{2\sigma^2}+2\frac{\sum \bar{y}_j\theta_j}{\sigma^2}-\frac{\sum\theta_j^2}{\sigma^2}-\frac{\sum\theta_j^2}{2\tau^2}\right\}$$

Completing the square in θ_j)

$$\frac{2\bar{y}_j}{\sigma^2}\theta_j-\frac{1}{\sigma^2}\theta_j^2-\frac{1}{2\tau^2}\theta_j^2 =$$

$$= -\frac{1}{2}\left(\frac{1}{\tau^2}+\frac{2}{\sigma^2}\right)\theta_j^2+\frac{2\bar{y}_j}{\sigma^2}\theta_j$$

$$= -\frac{1}{2}\left(\frac{1}{\tau^2}+\frac{2}{\sigma^2}\right)\left(\theta_j-\frac{2\bar{y}_j}{\sigma^2}\left(\frac{1}{\tau^2}+\frac{1}{\sigma^2}\right)^{-1}\right)^2+\frac{2\bar{y}_j^2}{\sigma^2}\left(\frac{1}{\tau^2}+\frac{1}{\sigma^2}\right)^{-1}$$

$$= -\frac{1}{2}\left(\frac{1}{\tau^2}+\frac{2}{\sigma^2}\right)\left(\theta_j-\frac{2\bar{y}_j}{\sigma^2}\left(\frac{1}{\tau^2}+\frac{2}{\sigma^2}\right)^{-1}\right)^2+\frac{2\bar{y}_j^2\tau^2}{\sigma^2+2\tau^2}$$

$$\therefore \int p(\theta,\sigma,\tau|y)\,d\theta \propto \sigma^{-2J}|\tau|^{-J} e^{-\frac{\sum y_{ij}^2}{2\sigma^2}} e^{\frac{2\sum\bar{y}_j^2\tau^2}{\sigma^2+2\tau^2}} \prod_{j=1}^{J}\left(\frac{1}{\tau^2}+\frac{2}{\sigma^2}\right)^{-1}$$

$$\propto \sigma^{-2J}|\tau|^{-J}\frac{1}{\left(\frac{\sigma^2+2\tau^2}{\sigma^2\tau^2}\right)^J} e^{-\frac{\sum y_{ij}^2}{2\sigma^2}} e^{\frac{\tau^2 2\sum\bar{y}_j^2}{\sigma^2+2\tau^2}}$$

$$\propto \frac{|\tau|^{J}}{(\sigma^2+2\tau^2)^J}\exp\left\{-\frac{\sum y_{ij}^2}{2\sigma^2}+2\frac{\tau^2\sum\bar{y}_j^2}{\sigma^2+2\tau^2}\right\}$$

$$\therefore \int p(\theta,\sigma,\tau|y)\,d\theta\,d\sigma\,d\tau = k\int \frac{|\tau|^J}{(\sigma^2+2\tau^2)^J}\exp\left\{-\frac{\sum Y_{ij}^2}{2\sigma^2}+\frac{2\tau^2\sum\bar{Y}_j^2}{\sigma^2+2\tau^2}\right\}d\sigma\,d\tau$$

$$=\tilde{k}\int\frac{|\tau|^J}{(\sigma^2+\tau^2)^J}\exp\left\{-\frac{\sum Y_{ij}^2}{2\sigma^2}+\frac{\tau^2\sum\bar{Y}_j^2}{\sigma^2+\cancel{2}\tau^2}\right\}d\sigma\,d\tau \qquad (\tau'=\cancel{2}\sqrt{2}\,\tau)$$

$$=\tilde{k}\int_0^{2\pi}\int_0^{\infty}\frac{r^J|\cos\phi|^J}{r^{2J}}\exp\left\{-\frac{\sum Y_{ij}^2}{2r^2\cos^2\phi}+\frac{r^2\cos^2\phi\sum\bar{Y}_j^2}{r^2}\right\}r\,dr\,d\phi \qquad (\sigma=r\sin\phi,\ \tau=r\cos\phi)$$

$$=\tilde{k}\int_0^{2\pi}\int_0^{\infty}\frac{1}{r^{J-1}}\exp\left\{-\frac{\sum Y_{ij}^2}{2r^2\sin^2\phi}\right\}dr\ |\cos\phi|^J\exp\left\{\cos^2\phi\sum\bar{Y}_j^2\right\}d\phi$$

$$\le\tilde{k}\int_0^{2\pi}\int_0^{\infty}\frac{1}{r^{J-1}}e\left\{-\frac{\frac{1}{2}\sum Y_{ij}^2}{r^2}\right\}dr\ |\cos\phi|^J\exp\left\{\cos^2\phi\sum\bar{Y}_j^2\right\}d\phi$$

~~= LHS~~ $<\infty$ if $J\geq 3$ □

(as $\int_0^{\infty}\frac{1}{r^{J-1}}\exp\left\{-\frac{\frac{1}{2}\sum Y_{ij}^2}{r^2}\right\}dr=\int_0^{\infty}r^{+\frac{1}{2}J-2}e^{-\frac{r}{2}\sum Y_{ij}^2}\,dr$ $\quad(\tilde{r}=\frac{1}{r^2}$

is finite for $J>2$ i.e. $J\geq 3$)

Also for $J=1$ or 2,

$$\text{LHS}\geq\tilde{k}\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon}\int_0^{\infty}\frac{1}{r^{J-1}}e^{-\frac{\sum Y_{ij}^2}{r^2}}dr\ |\cos\phi|^J\exp\left\{\cos^2\phi\sum\bar{Y}_j^2\right\}d\phi$$

$$=\tilde{k}\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon}\infty\ |\cos\phi|^J\exp\left\{\cos^2\phi\sum\bar{Y}_j^2\right\}d\phi$$

$$=\infty$$ □

2005 Q.4

$$p(\theta_1,\dots,\theta_J,\sigma,\tau \mid y) \propto L(\theta_1,\dots,\theta_J,\sigma^2;y)\,\pi(\theta_j\mid\tau^2)\cdot p(\sigma,\tau)$$

$$\propto \left\{\prod_{i=1}^{2}\prod_{j=1}^{J}(2\pi\sigma^2)^{-\frac{1}{2}}e^{-\frac{1}{2\sigma^2}(y_{ij}-\theta_j)^2}\right\}\left\{\prod_{j=1}^{J}(2\pi\tau^2)^{-\frac{1}{2}}e^{-\frac{1}{2\tau^2}\theta_j^2}\right\}$$

$$\propto \sigma^{-2J}\prod_{j=1}^{J}e^{-\frac{1}{2\sigma^2}2\left(\frac{y_{1j}+y_{2j}}{2}-\theta_j\right)^2}\cdot\tau^{-J}e^{-\frac{1}{2\tau^2}\sum\theta_j^2}$$

not quite

$$\propto \sigma^{-2J}|\tau|^{-J}\exp\left\{-\frac{1}{\sigma^2}\sum_{j=1}^{J}\left(\frac{y_{1j}+y_{2j}}{2}-\theta_j\right)^2-\frac{1}{2\tau^2}\sum\theta_j^2\right\}$$

The posterior is proper iff the integral of this expression w.r.t. $\theta_1,\theta_2,\dots,\theta_J,\tau,\sigma$ is finite. Writing $\bar{y}_j=\frac{y_{1j}+y_{2j}}{2}$,

$$-\frac{1}{\sigma^2}\sum_1^J(\bar{y}_j-\theta_j)^2-\frac{1}{2\tau^2}\sum_1^J\theta_j^2 =$$

$$\underline{\Rightarrow\Sigma} = -\frac{1}{2}\sum_{j=1}^{J}2\frac{\bar{y}_j^2}{\sigma^2}-4\frac{\bar{y}_j\theta_j}{\sigma^2}+2\frac{\theta_j^2}{\sigma^2}+\frac{\theta_j^2}{\tau^2}$$

$$=-\frac{1}{2}\sum_{j=1}^{J}\left(\frac{2}{\sigma^2}+\frac{1}{\tau^2}\right)\theta_j^2-\frac{4\bar{y}_j}{\sigma^2}\theta_j+2\frac{\bar{y}_j^2}{\sigma^2}$$

$$=-\frac{1}{2}\sum_{j=1}^{J}\left(\frac{2}{\sigma^2}+\frac{1}{\tau^2}\right)\left(\theta_j-\frac{2\bar{y}_j/\sigma^2}{\left(\frac{2}{\sigma^2}+\frac{1}{\tau^2}\right)}\right)^2+2\frac{\bar{y}_j^2}{\sigma^2}-\frac{4\bar{y}_j^2/\sigma^2}{\frac{2}{\sigma^2}+\frac{1}{\tau^2}}$$

$$=-\frac{1}{2}\sum_{j=1}^{J}\left[\left(\frac{2}{\sigma^2}+\frac{1}{\tau^2}\right)\left(\theta_j-\frac{2\bar{y}_j/\sigma^2}{\frac{2}{\sigma^2}+\frac{1}{\tau^2}}\right)^2\right]-\frac{\bar{y}_j^2}{\sigma^2}+\frac{2\bar{y}_j^2}{2\sigma+\frac{\sigma^2}{\tau^2}}$$

by completing the sq squares.

Integrating in θ_j, we find

$$p(\sigma,\tau\mid y)\propto\sigma^{-2J}|\tau|^{-J}\cdot\frac{1}{\left(\frac{2}{\sigma^2}+\frac{1}{\tau^2}\right)^J}e^{-\frac{\sum\bar{y}_j^2}{\sigma^2}+\frac{\sum\bar{y}_j^2}{1+\sigma^2/2\tau^2}}$$

$$\propto \frac{|\tau|^{-J}}{(2\tau^2+\sigma^2)^J}\, e^{-\frac{\sum \bar{y}_j^2}{\sigma^2}}\, e^{\frac{2\tau^2 \sum \bar{y}_j^2}{2\tau^2+\sigma^2}} \qquad \text{Ⓘ}$$

Alternatively, we can integrate in σ and τ first.

~~$\int p(\theta_1,\dots,\theta_J,\sigma,\tau|y)\,d\sigma = \int p(\theta_1,\dots,\theta_J,\sqrt{t},\tau|y)\,\frac{1}{2}t^{-\frac{1}{2}}\,dt \qquad (t=\sigma^2)$~~

~~$=\int$~~ $\qquad \left(s = \frac{1}{\sigma^2}\ ;\ \frac{d\sigma}{ds} = -\frac{1}{2}s^{-\frac{3}{2}}\right.$

$$\int_{-\infty}^{\infty} p(\theta_1,\dots,\theta_J,\sigma,\tau|y)\,d\sigma = 2\int_0^{\infty} p\left(\theta_1,\dots,\theta_J,\frac{1}{\sqrt{s}},\tau|y\right)\frac{1}{2}s^{-\frac{3}{2}}\,ds$$

$$= \int_0^{\infty} s^{J-\frac{3}{2}} \exp\left\{-\left(\sum_{j=1}^{J}(\bar{y}_j-\theta_j)^2\right)s\right\} ds\, |\tau|^{-J} e^{-\frac{\sum\theta_j^2}{2\tau^2}}$$

$$= \frac{\Gamma(J-\frac{1}{2})}{\left(\sum_1^J(\bar{y}_j-\theta_j)^2\right)^{J-\frac{1}{2}}}\,|\tau|^{-J} e^{-\frac{\sum\theta_j^2}{2\tau^2}} \qquad \text{(Gamma density)}$$

Similarly we can integrate out τ by letting $t = \frac{1}{\tau^2}$

$$\therefore\ p(\theta_1,\dots,\theta_J|y) = \iint p(\theta_1,\dots,\theta_J,\sigma,\tau|y)\,d\sigma\,d\tau$$

$$\propto \frac{1}{\left(\sum(\bar{y}_j-\theta_j)^2\right)^{J-\frac{1}{2}}}\cdot\int_0^{\infty} t^{\frac{J}{2}} e^{-\frac{\sum\theta_j^2}{2}t}\, t^{-\frac{3}{2}}\,dt$$

$$\propto \frac{1}{\left(\sum(\bar{y}_j-\theta_j)^2\right)^{J-\frac{1}{2}}}\int_0^{\infty} t^{\left(\frac{J-1}{2}\right)-1} e^{-\left(\frac{\sum\theta_j^2}{2}\right)t}\,dt$$

$$\propto \frac{1}{\left(\sum(\bar{y}_j-\theta_j)^2\right)^{J-\frac{1}{2}}}\cdot\frac{1}{\left(\frac{\sum\theta_j^2}{2}\right)^{\frac{J-1}{2}}} \qquad \text{if } J\geq 2$$

∞ otherwise

2005 Q4

From I, note (scaling out τ by $\sqrt{2}$)

$$\iint p(\sigma,\tau|y)\, d\sigma\, d\tau = k \iint \frac{|\tau|^J}{(\tau^2+\sigma^2)^J} e^{-\frac{\Sigma y_j^2}{\sigma^2} + \frac{\tau^2 \Sigma \bar{y}_j^2}{\tau^2+\sigma^2}} d\tau\, d\sigma$$

$$= k \int_0^{2\pi}\int_0^{\infty} \frac{r^J |\cos\phi|^J}{r^{2J}} e^{-\frac{\Sigma \bar{y}_j^2}{r^2 \sin^2\phi} + \frac{r^2\cos^2\phi\, \Sigma\bar{y}_j^2}{r^2}} |r|\, dr\, d\phi$$

(let $\tau = r\cos\phi$, $\sigma = r\sin\phi$) $\left|\frac{\partial(\tau,\sigma)}{\partial(r,\phi)}\right| = |r|$)

$$= k \int_0^{2\pi} \int_0^{\infty} \frac{1}{r^{J-1}} \exp\left\{-\frac{\Sigma\bar{y}_j^2}{r^2\sin^2\phi}\right\} dr\, |\cos^J\phi|\, e^{\cos^2\phi\, \Sigma\bar{y}_j^2}\, d\phi$$

$$\leq k \int_0^{2\pi}\int_0^{\infty} \frac{e^{-\frac{\Sigma\bar{y}_j^2}{r^2}}}{r^{J-1}} dr\, \cos^J\phi\, e^{\cos^2\phi\, \Sigma\bar{y}_j^2}\, d\phi$$

$< \infty$ for $J \geq 3$ □

Also, $$\int_0^{2\pi}\int_0^{\infty} \frac{1}{r^{J-1}} \exp\left\{-\frac{\Sigma\bar{y}_j^2}{r^2\sin^2\phi}\right\} dr\, |\cos^J\phi|\, e^{\cos^2\phi\, \Sigma\bar{y}_j^2}\, d\phi$$

$$\geq \int_{\frac{1}{2}\pi-\varepsilon}^{\frac{1}{2}\pi+\varepsilon} \int_0^{\infty} \frac{1}{r^{J-1}} \exp\left\{\frac{-\Sigma\bar{y}_j^2}{r^2\sin^2\phi}\right\} dr\, |\cos^J\phi|\, d\phi$$

$$\geq \int_{\frac{1}{2}\pi-\varepsilon}^{\frac{1}{2}\pi+\varepsilon} \int_0^{\infty} \frac{1}{r^{J-1}} e^{-\frac{2\Sigma\bar{y}_j^2}{r^2}} dr\, |\cos^J\phi|\, d\phi$$

$$= \int_{\frac{1}{2}\pi-\varepsilon}^{\frac{1}{2}\pi+\varepsilon} \{\infty\} |\cos^J\phi|\, d\phi = \infty$$ for $J = 1$ or 2 □

2008 Q6

$$L(\theta;X) = p_\theta(X) = \prod_{i=1}^{n} P_\theta(X_i = x_i)$$

$$= \prod_{i=1}^{n} P_\theta\left(\text{Geom}(p) = \frac{x_i}{\theta}\right)$$

$$= \prod_{i=1}^{n} (1-p)\, p^{\frac{x_i}{\theta}}\, \mathbb{1}\{\tfrac{x_i}{\theta} \in \mathbb{Z}\}$$

$$= (1-p)^n\, p^{\frac{\sum x_i}{\theta}}\, \mathbb{1}\{\tfrac{x_i}{\theta} \in \mathbb{Z}\ \forall i\}$$

$$\therefore \quad \frac{L(\theta;X)}{L(\theta;Y)} = p^{\frac{\sum x_i - y_i}{\theta}}\, \mathbb{1}\{\tfrac{x_i}{\theta} \in \mathbb{Z}\ \forall i\} \Big/ \mathbb{1}\{\tfrac{y_i}{\theta} \in \mathbb{Z}\ \forall i\}$$

which is independent of θ iff $\sum x_i = \sum y_i$ and $\gcd(x_i) = \gcd(y_i)$

$\therefore$ M.S. statistic is $T = (\sum X_i, \gcd(X_i))$

2004 Q2

(i) $L(\sigma^2, \lambda_i, \beta; x, y) = (2\pi\sigma^2)^{-n} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_i^n (X_i - \lambda_i)^2 + (Y_i - \lambda_i - \beta W_i)^2\right]\right\}$

$\therefore\ l(\sigma^2, \lambda_i, \beta; x, y) = -n\log\sigma^2 - \frac{1}{2\sigma^2}\left[\Sigma(X_i - \lambda_i)^2 + (Y_i - \lambda_i - \beta W_i)^2\right]$ ④

To maximize l, we first minimize the quadratic term

$$Q(\lambda_i, \beta) = \Sigma(X_i - \lambda_i)^2 + \Sigma(Y_i - \lambda_i - \beta W_i)^2$$

$$\frac{\partial Q}{\partial \lambda_i} = -2(X_i - \lambda_i) - 2(Y_i - \lambda_i - \beta W_i)$$

$$\frac{\partial Q}{\partial \beta} = -2\Sigma W_i (Y_i - \lambda_i - \beta W_i)$$

$$\frac{\partial^2 Q}{\partial \lambda_i^2} = 4 \qquad \frac{\partial^2 Q}{\partial \beta^2} = 2\Sigma W_i^2 \qquad \frac{\partial^2 Q}{\partial \beta \partial \lambda_i} = 2W_i \qquad \frac{\partial^2 Q}{\partial \lambda_i \partial \lambda_j} = 0.$$

The Hessian matrix is therefore

$$H = \begin{pmatrix} 2\sum_i^n W_i^2 & 2W_1 & \cdots & 2W_n \\ 2W_1 & 4 & & 0 \\ \vdots & & \ddots & \\ 2W_n & 0 & & 4 \end{pmatrix}$$

Note that this is +ve definite:

$$x^T H x = x^T \begin{pmatrix} 2\Sigma W_i^2 x_1 + 2W_1 x_2 + \cdots + 2W_n x_{n+1} \\ 2\cancel{\Sigma} W_1 x_1 + \cancel{2}\, 4x_2 \\ 2W_2 x_1 \quad + 4x_3 \\ \vdots \\ 2W_n x_1 + 4x_{n+1} \end{pmatrix}$$

$$= 2\Sigma W_i^2 x_1^2 + 2W_1 x_1 x_2 + \cdots + 2W_n x_1 x_{n+1}$$
$$+ 2W_1 x_1 x_2 + 4x_2^2 + 2W_2 x_1 x_3 + 4x_3^2 + \cdots + 2W_n x_1 x_{n+1} + 4x_{n+1}^2$$

$$= 2\Sigma W_i^2 x_1^2 + 4\cancel{W}\, x_1 \Sigma x_i W_i + 4\Sigma x_i^2$$

$$= 2\Sigma (W_i x_1 + x_i)^2 + 2\Sigma x_i^2 > 0.$$

$\therefore$ the stationary point maximises l.

$$\therefore \quad X_i - \hat{\lambda}_i + Y_i - \hat{\lambda}_i - \hat{\beta} W_i = 0 \qquad \forall_i \qquad \text{(I)}$$

and $$\underbrace{\sum W_i (Y_i - \hat{\lambda}_i - \hat{\beta} W_i) = 0}_{} \qquad \text{(II)}$$

$$\hat{\beta} \sum W_i^2 = \sum W_i Y_i - \sum W_i \hat{\lambda}_i \qquad \text{(III)}$$

from I, $$\sum W_i X_i - 2 \sum W_i \hat{\lambda}_i + \sum W_i Y_i - \hat{\beta} \sum W_i^2 = 0$$

$$\therefore \quad \sum W_i \hat{\lambda}_i = \frac{1}{2}\left[\sum W_i (X_i + Y_i) - \hat{\beta} \sum W_i^2 \right]$$

plugging this into III,

$$\hat{\beta} = \frac{1}{\sum W_i^2} \left[\sum W_i Y_i - \frac{1}{2} \sum W_i (X_i + Y_i) + \frac{1}{2} \hat{\beta} \sum W_i^2 \right]$$

$$\therefore \quad \hat{\beta} = \frac{2}{\sum W_i^2} \left[\frac{1}{2} \sum W_i (Y_i - X_i) \right] = \frac{\sum W_i (Y_i - X_i)}{\sum W_i^2}$$

and ~~λ_i~~ from I, $$\hat{\lambda}_i = \frac{1}{2}\left[X_i + Y_i - W_i \hat{\beta} \right] = \frac{1}{2}\left[X_i + Y_i - W_i \frac{\sum W_j (Y_j - X_j)}{\sum W_j^2} \right]$$

~~For~~ Now, from IV, $l \to -\infty$ as $\sigma \to 0$ or ∞,

and $$\frac{\partial l}{\partial \sigma^2} = -\frac{n}{\sigma^2} + \frac{Q(\lambda_i, \beta)}{2(\sigma^2)^2}$$ which has a unique root at

$$\sigma^2 = \frac{Q(\lambda_i, \beta)}{2n} \qquad \therefore \text{ the MLE is}$$

$$\hat{\sigma}^2 = \frac{Q(\hat{\lambda}_i, \hat{\beta})}{2n} = \frac{\sum (X_i - \hat{\lambda}_i)^2 + (Y_i - \hat{\lambda}_i - \hat{\beta} W_i)^2}{2n}$$

~~$= \sum (X_i$~~

$$= \frac{1}{2n} \left[\sum \left(\frac{X_i - Y_i}{2} + W_i \frac{\sum W_j (Y_j - X_j)}{2 \sum W_j^2} \right)^2 + \sum \left(\frac{Y_i - X_i}{2} + \frac{W_i \sum W_j (Y_j - X_j)}{2 \sum W_j^2} \right. \right.$$

$$\left. \left. - \frac{W_i \sum W_j (Y_j - X_j)}{\sum W_j^2} \right) \right]$$

2004 Q2

$$= \frac{1}{2n}\left[\sum\left(\frac{X_i-Y_i}{2} + W_i\frac{\sum W_j(Y_j-X_j)}{2\sum W_j^2}\right)^2 + \sum\left(\frac{Y_i-X_i}{2} - \frac{W_i\sum W_j(Y_j-X_j)}{2\sum W_j^2}\right)^2\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}\left(\frac{Y_i-X_i}{2} - \frac{W_i\sum W_j(Y_j-X_j)}{2\sum W_j^2}\right)^2\right]$$

$$= \frac{1}{4n}\left[\sum_{1}^{n}\left(Y_i-X_i - W_i\frac{\sum W_j(Y_j-X_j)}{\sum W_j^2}\right)^2\right]$$

$$= \frac{1}{4n}\left[\sum_{1}^{n}\left(Y_i-X_i-\beta W_i - \left\{W_i\frac{\sum W_j(Y_j-X_j)}{\sum W_j^2} - \beta W_i\right\}\right)^2\right]$$

$$= \frac{1}{4n}\left[\sum_{1}^{n}\left\{(Y_i-X_i-\beta W_i)^2 - 2(Y_i-X_i-\beta W_i)W_i\left(\frac{\sum W_j(Y_j-X_j)}{\sum W_j^2}-\beta\right) + W_i^2\left(\frac{\sum W_j(Y_j-X_j)}{\sum W_j^2}-\beta\right)^2\right\}\right]$$

$$= \frac{1}{4n}\left[\sum_{1}^{n}(Y_i-X_i-\beta W_i)^2 - 2\left(\sum W_i(Y_i-X_i) - \beta\sum W_i^2\right)\left(\frac{\sum W_j(Y_j-X_j)}{\sum W_j^2}-\beta\right) + \sum W_i^2\left(\frac{\sum W_j(Y_j-X_j)}{\sum W_j^2}-\beta\right)^2\right]$$

$$= \frac{1}{4n}\left[\sum_{1}^{n}(Y_i-X_i-\beta W_i)^2 - (\sum W_j^2)\left(\frac{\sum W_j(Y_j-X_j)}{\sum W_j^2}-\beta\right)^2\right]$$

$$= \frac{\sum(Y_i-X_i-\beta W_i)^2}{4n} - \cancel{\beta}\frac{(\sum W_j^2)}{4n}\left(\frac{\sum W_j(Y_j-X_j)}{\cancel{\beta}\sum W_j^2} - \frac{\beta}{\cancel{\beta}}\right)^2$$

$\cancel{\Sigma(Y_i - X_i - \beta W_i)^2}$ $\quad Y_i - X_i - \beta W_i \overset{iid}{\sim} N(0, 2\sigma^2) \;\therefore\; E\,\frac{\Sigma(Y_i-X_i-\beta W_i)^2}{4n} =$

Now note

$$\therefore\; \frac{\sum(Y_i-X_i-\beta W_i)^2}{4n} \xrightarrow{P} \frac{\sigma^2}{2} \text{ by WLLN.}$$

Also $\cancel{\beta}\,\frac{1}{4n}(\sum W_j^2)\left(\frac{\sum W_j(Y_j-X_j)}{\cancel{\beta}\,\sum W_j^2} - \frac{\beta}{\cancel{\beta}}\right)^2 =$

$$\cancel{= \frac{\sum W_j}{\ } \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\sum W_j(Y_j-X_j) - \frac{1}{\sqrt{n}}\beta\sum W_j^2\right)^2}$$

~~$= \left(\frac{1}{\sqrt{n}} \Sigma W_i (Y_i - X_i - W_i \beta) \right)^2$~~

~~$\therefore \frac{1}{\sqrt{n}}$ But $W_i(Y_i - X_i - W_i\beta)$~~

$$= \frac{1}{4n \Sigma W_j^2} \left(\Sigma W_i (Y_i - X_i - W_i \beta) \right)^2$$

But ~~$W_i(X_i$~~ $W_i(Y_i - X_i - W_i\beta) \sim N(0, 2\sigma^2 W_i^2)$

~~$\therefore E \frac{1}{4n \Sigma W_j^2}$~~ $\therefore \quad \Sigma W_i (Y_i - X_i - W_i \beta) \sim N(0, 2\sigma^2 \Sigma W_i^2)$

$$\therefore \quad E \left(\Sigma W_i (Y_i - X_i - W_i \beta) \right)^2 = 2\sigma^2 \Sigma W_i^2$$

$$\therefore \quad E \left[\frac{1}{4n \Sigma W_j} \left(\Sigma W_i (Y_i - X_i - W_i \beta) \right)^2 \right] = \frac{\sigma^2}{2n}$$

Putting the pieces together,

$$E \hat{\sigma}^2_{MLE} = \frac{\sigma^2}{2} - \frac{\sigma^2}{2n} = \frac{\sigma^2}{2} \left(\frac{n-1}{n} \right) \longrightarrow \frac{\sigma^2}{2} \text{ as } n \to \infty$$

$\therefore$ MLE not asymptotically unbiased □

(ii) Observations are not iid.

2004 Q3

(i) $L(\lambda_i, \beta; X, Y) = \prod_{i=1}^{n} \frac{e^{-e^{\lambda_i}} e^{\lambda_i X_i}}{X_i!} \cdot \frac{e^{-e^{\lambda_i+\beta W_i}} e^{(\lambda_i+\beta W_i)Y_i}}{Y_i!}$

$\therefore \quad \ell(\lambda_i; \beta; X, Y) = \sum_{i=1}^{n} -e^{\lambda_i} \cancel{+} - e^{\lambda_i+\beta W_i} + \lambda_i X_i + \lambda_i Y_i + \beta W_i Y_i$

$\therefore \quad \frac{\partial \ell}{\partial \beta} = \sum^{n} \cancel{*} - W_i e^{\lambda_i+\beta W_i} + W_i Y_i$

$\frac{\partial \ell}{\partial \lambda_i} = -e^{\lambda_i} \cancel{+} - e^{\lambda_i+\beta W_i} + X_i + Y_i$

$\frac{\partial^2 \ell}{\partial \beta^2} = -\sum W_i^2 e^{+\lambda_i+\beta W_i} \qquad \frac{\partial^2 \ell}{\partial \lambda_i} = -e^{\lambda_i} \cancel{+} - e^{\lambda_i+\beta W} \qquad \frac{\partial^2 \ell}{\partial \lambda_i \partial \beta} = -W_i e^{\lambda_i+\beta W_i}$

<u>Claim</u>: the Hessian is ~~+ve~~ -ve definite.

$\frac{\partial \ell}{\partial \lambda_i} = 0 \implies e^{\lambda_i}(1 + e^{\beta W_i}) = X_i + Y_i \quad \therefore e^{\lambda_i} = \frac{X_i + Y_i}{1 + e^{\beta W_i}}$

$\therefore \quad \hat{\lambda}_i = \log\left(\frac{X_i + Y_i}{1 + e^{\beta W_i}}\right)$

$\frac{\partial \ell}{\partial \beta} = 0 \implies -\sum W_i e^{\lambda_i} e^{\beta W_i} + W_i Y_i = 0$

$\implies -\sum W_i (X_i + Y_i) \frac{e^{\beta W_i}}{1 + e^{\beta W_i}} + W_i Y_i = 0$

$\implies \sum W_i (X_i + Y_i) \frac{e^{\beta W_i}}{1 + e^{\beta W_i}} = \sum W_i Y_i$

$\ell(\beta, \hat{\lambda}_i; X, Y) = \sum_{i=1}^{n} -\frac{X_i + Y_i}{1 + e^{\beta W_i}} - \frac{X_i + Y_i}{1 + e^{\beta W_i}} e^{\beta W_i} + X_i \log \frac{X_i + Y_i}{1 + e^{\beta W_i}} + Y_i \log \frac{X_i + Y_i}{1 + e^{\beta W_i}}$

$+ \beta W_i Y_i$.

~~$\ell'(\beta) = -\sum$~~

$$l'(\beta) = -\sum \frac{X_i+Y_i}{(1+e^{\beta W_i})^2}$$

$$1 - \frac{1}{1+e^{\beta W_i}}$$

$$\frac{\partial l}{\partial \beta}(\beta,\hat{\lambda}_i;X,Y) = \cancel{-\sum W_i e^{\lambda_i+\beta W_i} \quad +} \sum W_i Y_i$$

$$= -\sum W_i (X_i+Y_i) \frac{e^{\beta W_i}}{1+e^{\beta W_i}} + \sum W_i Y_i$$

~~$\frac{\partial^2 l}{\partial \beta^2}(\beta,\hat{\lambda}_i;X,Y) =$~~ Differentiating this in β,

$$\frac{\partial^2 l}{\partial \beta^2} \quad \frac{d}{d\beta}\left(\frac{\partial l}{\partial \beta}(\beta,\hat{\lambda}_i;X,Y)\right) = -\sum \frac{W_i^2(X_i+Y_i)}{(1+e^{\beta W_i})^2}$$

By Taylor's thm, (assuming regularity conditions)

$$0 = -\sum W_i(X_i+Y_i)\frac{e^{\beta W_i}}{1+e^{\beta W_i}} + \sum W_i Y_i + (\hat{\beta}-\beta)\left(-\sum \frac{W_i^2(X_i+Y_i)}{(1+e^{\beta W_i})^2}\right) + \overset{0}{\cancel{\text{h.o.t.}}}$$

$$\therefore \quad (\hat{\beta}-\beta) = \frac{-\sum W_i(X_i+Y_i)\frac{e^{\beta W_i}}{1+e^{\beta W_i}} + W_i Y_i}{\sum\left(\frac{W_i^2(X_i+Y_i)}{(1+e^{\beta W_i})^2}\right)}$$

Now note

$$E\ -W_i(X_i+Y_i)\frac{e^{\beta W_i}}{1+e^{\beta W_i}} + W_i Y_i = W_i\left[-e^{\lambda_i}e^{\beta W_i} + e^{\lambda_i+\beta W_i}\right] = 0$$

$$\text{Var}\left(-W_i(X_i+Y_i)\frac{e^{\beta W_i}}{1+e^{\beta W_i}} + W_i Y_i\right) = \text{Var}\left(W_i Y_i\left(1-\frac{e^{\beta W_i}}{1+e^{\beta W_i}}\right) - W_i X_i \frac{e^{\beta W_i}}{1+e^{\beta W_i}}\right)$$

$$= \frac{W_i^2}{(1+e^{\beta W_i})^2} e^{\lambda_i+\beta W_i} + \frac{W_i^2 e^{2\beta W_i}}{(1+e^{\beta W_i})^2} e^{\lambda_i}$$

$$= \frac{W_i^2 e^{\lambda_i} e^{\beta W_i}}{(1+e^{\beta W_i})^2}(1+e^{\beta W_i}) = \frac{W_i^2 e^{\lambda_i} e^{\beta W_i}}{(1+e^{\beta W_i})}$$

~~Let $\sigma_i^2 = W_i^2 e^{\lambda_i}$~~

2004 Q3

But $E\left(\frac{W_i^2(X_i+Y_i)}{(1+e^{\beta W_i})^2}\right) = \frac{W_i^2 e^{\lambda_i}(1+e^{\beta W_i})}{(1+e^{\beta W_i})^2}$

$$= \frac{W_i^2 e^{\lambda_i}}{1+e^{\beta W_i}}$$

∴ By regularity conditions / ~~[illegible]~~ Lyapunov CLT β

$$\underline{E} \qquad \frac{\sum W_i Y_i - W_i(X_i+Y_i)\frac{e^{\beta W_i}}{1+e^{\beta W_i}}}{\sqrt{\sum \frac{W_i^2 e^{\lambda_i} e^{\beta W_i}}{1+e^{\beta W_i}}}} \xrightarrow{d} N(0,1)$$

$$\frac{\sum \frac{W_i^2(X_i+Y_i)}{(1+e^{\beta W_i})^2}}{\sum \frac{W_i^2 e^{\lambda_i}}{1+e^{\beta W_i}}} \xrightarrow{P} 1$$

∴ $\underline{\sum W_i^2 e^{\lambda_i}}$

$$\therefore \frac{\sum \frac{W_i^2 e^{\lambda_i}}{1+e^{\beta W_i}}}{\sqrt{\sum \frac{W_i^2 e^{\lambda_i} e^{\beta W_i}}{1+e^{\beta W_i}}}}(\hat{\beta}-\beta) \longrightarrow N(0,1)$$

Assume $\frac{\sum \frac{W_i^2 e^{\lambda_i}}{1+e^{\beta W_i}}}{\sqrt{\sum \frac{W_i^2 e^{\lambda_i} e^{\beta W_i}}{1+e^{\beta W_i}}}} \sim \sqrt{n}$ as $n \to \infty$

2004 Q4

$P_{\lambda=\frac{1}{2}, \beta=\frac{1}{2}} \left(-2\log\Lambda > z^2_{0.975} \right)$

$\frac{h}{\sqrt{n}}$ use contiguity

$(\lambda, \beta) = \left(\frac{5}{\sqrt{100}}, \frac{5}{\sqrt{100}} \right)$

$-2\log\Lambda \xrightarrow[H_0]{d} \chi^2_2$

manifold of dimension p_0

$-2\log\Lambda \xrightarrow[H_0]{d} \chi^2_{p_1-p_0}$ $H_0 : \theta \in \Theta_0$

$H_1 : \theta \in \Theta_1$ ← manifold of dimension p_1.

2004 Q5

(i) ~~$L(\alpha,\beta$~~

$$p_{\alpha,\beta}(X,Y) = p_{\alpha,\beta}(Y|X)\, p_{\alpha,\beta}(X)$$

$$= \prod_{i=1}^{n} \left(\frac{1}{1+\alpha\sin(\beta x_i)}\right)^{Y_i} \left(\frac{\alpha\sin(\beta x_i)}{1+\alpha\sin(\beta x_i)}\right)^{1-Y_i} f_X(X_i)$$

$$p_{\alpha,\beta}(Y) = \int p_{\alpha,\beta}(X,Y)\, dX$$

$$p_{\alpha,\beta}(Y_i) = \int \left(\frac{1}{1+\alpha\sin(\beta x)}\right)^{Y_i} \left(\frac{\alpha\sin(\beta x)}{1+\alpha\sin(\beta x)}\right)^{1-Y_i} ~~dx~~\; f(x)\, dx$$

$$= \begin{cases} \int \frac{1}{1+\alpha\sin\beta x} f(x)\, dx & \text{if } Y_i = 1 \\ \int \frac{\alpha\sin(\beta x)}{1+\alpha\sin\beta x} f(x)\, dx & \text{if } Y_i = 0 \end{cases}$$

$$= \begin{cases} \int \frac{1}{1+\alpha\sin x} \frac{f(x/\beta)}{\beta}\, dx & \text{if } Y_i = 1 \\ \int \frac{\alpha\sin x}{1+\alpha\sin x} \frac{f(x/\beta)}{\beta}\, dx & \text{if } Y_i = 0 \end{cases}$$

$$= \begin{cases} E_{X\sim\tilde{f}}\left[\frac{1}{1+\alpha\sin X}\right] & \text{if } Y_i = 1 \\ E_{X\sim\tilde{f}}\left[\frac{\sin X}{1+\alpha\sin X}\right] & \text{if } Y_i = 0 \end{cases}$$

where $\tilde{f}(t) = \frac{f(t/\beta)}{\beta}$

as the density of X is unspecified,

this is independent of β, as required

2004 Q7

The likelihood is

~~$L(p_{00}, p_{01}, p_{10}$~~

$$L(p_{01}, p_{10}; X) = P(X_1, X_2, \dots, X_n)$$

$$= P(X_n | X_{n-1}, \dots, X_1) P(X_{n-1} | X_{n-2}, \dots, X_1) \cdots P(X_2|X_1) P(X_1)$$

$$= P(X_n|X_{n-1}) P(X_{n-1}|X_{n-2}) \cdots P(X_2|X_1) P(X_1)$$

$$= p_{00}^{n_{00}} \; p_{01}^{n_{01}} \; p_{10}^{n_{10}} \; p_{11}^{n_{11}} \; P(X_1)$$

where n_{ij} denotes the number of steps from i to j,
$(n_{00} + n_{01} + n_{10} + n_{11} = n)$

and $p_{00} = 1 - p_{01}$, $p_{11} = 1 - p_{10}$.

$$\therefore \; L(p_{01}, p_{10}; X) = (1-p_{01})^{n_{00}} \; p_{01}^{n_{01}} \; (1-p_{10})^{n_{11}} \; p_{10}^{n_{10}}$$

~~Fix an alternative = exp{~~

Fix an alternative $p_{10}^{(1)} > p_{01}^{(1)}$.

For a least favourable prior, put mass 1 on

$$p_{10} = p_{01} = p \in (p_{01}^{(1)}, p_{10}^{(1)})$$

$$\frac{L(p_{01}^{(1)}, p_{10}^{(1)}; X)}{L(p, p; X)} = \underbrace{\left(\frac{1-p_{01}^{(1)}}{1-p}\right)^{n_{00}}}_{>1} \underbrace{\left(\frac{p_{01}^{(1)}}{p}\right)^{n_{01}}}_{<1} \underbrace{\left(\frac{1-p_{10}^{(1)}}{1-p}\right)^{n_{11}}}_{<1} \underbrace{\left(\frac{p_{10}^{(1)}}{p}\right)^{n_{10}}}_{>1}$$

$\doteq \exp \{$

2001 Q1

$$L(\theta; X) = \prod_{i=1}^{n} \frac{1}{3} \mathbb{1}\{x_i \in \{\theta-1, \theta, \theta+1\}\}$$

$$= \frac{1}{3^n} \mathbb{1}\{x_{(1)} \geq \theta-1\} \mathbb{1}\{x_{(n)} \leq \theta+1\} \mathbb{1}\{x_i \in \mathbb{Z}\ \forall i\}$$

Thus $\dfrac{L(\theta;X)}{L(\theta;Y)} = \dfrac{\mathbb{1}\{x_{(1)} \geq \theta-1\} \mathbb{1}\{x_{(n)} \leq \theta+1\}}{\mathbb{1}\{y_{(1)} \geq \theta-1\} \mathbb{1}\{y_{(n)} \leq \theta+1\}}$

Clearly this is $\perp\!\!\!\perp$ of θ iff $(X_{(1)}, X_{(n)}) = (Y_{(1)}, Y_{(n)})$

$\therefore$ $T(x) = (X_{(1)}, X_{(n)})$ is M.S.

~~By the~~ Now assume, for a contradiction, that $\exists$ a C.S. statistic. By class results, then T is C.S.

Compute $P_\theta(X_{(1)} = \theta+1) = \frac{1}{3^n}$

$$P_\theta(X_{(1)} = \theta) = P_\theta(X_{(1)} \geq \theta) - P_\theta(X_{(1)} = \theta+1) = \left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n$$

$$P_\theta(X_{(1)} = \theta-1) = 1 - P(X_{(1)} \geq \theta) = 1 - \left(\frac{2}{3}\right)^n$$

~~Thus~~, Similarly,

$P_\theta(X_{(n)} = \theta-1) = \left(\frac{1}{3}\right)^n$, $P_\theta(X_{(n)} = \theta) = \left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n$, $P_\theta(X_{(n)} = \theta+1) = 1 - \left(\frac{2}{3}\right)^n$.

$$\therefore E X_{(1)} = \left(1 - \left(\tfrac{2}{3}\right)^n\right)(\theta-1) + \theta\left(\left(\tfrac{2}{3}\right)^n - \left(\tfrac{1}{3}\right)^n\right) + (\theta+1)\left(\tfrac{1}{3^n}\right)$$
$$= \theta + \frac{1}{3^n} - \left(\frac{2}{3}\right)^n$$

Similarly, $E X_{(n)} = \theta + \left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n$

Thus, $E_\theta\left[X_{(n)} - X_{(1)} - 2\left(\frac{2}{3}\right)^n + 2\left(\frac{1}{3}\right)^n\right] = 0 \quad \forall\theta$

$\therefore$ $T(X) = (X_{(1)}, X_{(n)})$ is not C.S. X

2001 Q2

Let $A = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_p \end{pmatrix}$

To begin with, apply Gram-Schmidt to the vectors $\vec{a}_1, \dots, \vec{a}_p$

to obtain $\tilde{a}_1 = \frac{a_1}{\|a_1\|}$, $\tilde{a}_2 = \frac{a_2 - (\tilde{a}_1^T a_2)\tilde{a}_1}{\|a_2 - (\tilde{a}_1^T a_2)\tilde{a}_1\|}$, ...

so that $\tilde{A} = \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_p \end{pmatrix}$ has orthonormal rows, and $A\mu = 0 \iff \tilde{A}\mu = 0$.

extend ~~$\tilde{A}$ to~~ $\tilde{a}_1, \dots, \tilde{a}_p$ to an o/n basis of $\mathbb{R}^k$,

$\tilde{a}_1, \dots, \tilde{a}_k$. Let ~~$\tilde{A}_{k\times k} = \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_k \end{pmatrix}$~~, $P = \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_k \end{pmatrix}$, so

~~Then $\tilde{A}_{k\times k}$~~ that P is ~~often~~ orthogonal.

Let $Y_i = PX_i$. Then $Y_i \overset{iid}{\sim} N(P\mu, I_k)$,

and ~~[illegible]~~ Y is a 1-1 map of our data.

Our null is equivalent to $EY_i = 0$ for $i = 1, \dots, p$.

As the components are independent, we can ignore the last

$k-p$ components of the observations $\vec{Y}_i$ (can formalise

this using least favourable prior). Writing $\tilde{\mu} = P\mu$,

have $Y_i \overset{iid}{\sim} N(\tilde{\mu}, I_k)$ and ~~can therefore reduce to testing~~

our test reduces to $\mu_1 = \cdots = \mu_p = 0$.

Case 1: $p = 1$. By class results, UMPU is given by

$$\phi_1(Y) = 1 \quad \text{if} \quad |\sqrt{n}\,\bar{Y}_1| > z_{1-\frac{\alpha}{2}}$$
$$= 0 \quad \text{o/w}$$

i.e. $\phi_1(X) = 1$ if $|\sqrt{n}\ \bar{Y}_1(X)| > z_{1-\frac{\alpha}{2}}$.

Case 2: $p \geq 2$. Then no UMPU test exists.

Consider the alternatives: ~~$\mu =$~~ I : $\mu_1 = 1, \mu_2 = 0,$ ~~Ξ~~ $(H_1^{(1)})$
II : $\mu_1 = 0, \mu_2 = 1.$ $(H_1^{(2)})$

By case 1, ϕ_1 achieves maximal power $\overset{\beta_1}{\text{at}}$ $\overset{H_1^{(1)}}{\text{I}}$ of all level α unbiased Tests. Similarly, $\phi_2(Y) := 1$ iff $|\sqrt{n}\,\bar{Y}_2| > z_{1-\alpha/2}$ achieves maximal power $\overset{\beta_2 = \beta_1}{\text{at}}$ $H_1^{(2)}$ of all level α unbiased test. However, if ψ is unbiased ~~for $H: \mu_1 = 0, \mu_2 = 0$,~~ level α, it cannot satisfy $E_{H_1^{(1)}}\psi(X) = E_{H_1^{(2)}}\psi(X) = \beta_1$ $\dot{X}$
To see this note that if ψ is unbiased level α and $E_{H_1^{(1)}}\psi(X) = \beta_1$, then ψ is UMPU for case 1,
$\therefore$ ψ reject if $|\sqrt{n}\,\bar{Y}_1| > z_{1-\frac{\alpha}{2}}$.
~~Since~~ But then, ψ level α $\Longrightarrow$ ~~E~~ $\psi = 0$ if $|\sqrt{n}\,\bar{Y}_1| < z_{1-\frac{\alpha}{2}}$.
$\therefore$ $E_{H_1^{(2)}}\psi < \beta_1$ $\square$.

2001 Q3

$$L(\alpha_i, \theta, \sigma^2 ; X, Y) = (2\pi\sigma^2)^{-n} \exp\left\{ -\frac{1}{2\sigma^2} \Sigma\left[(X_i - \alpha_i)^2 + (Y_i - \alpha_i - \theta)^2 \right] \right\}$$

$$\therefore \ l(\alpha_i, \theta, \sigma^2 ; X, Y) = -n \log \sigma^2 - \frac{1}{2\sigma^2}\left\{ \Sigma\ (X_i - \alpha_i)^2 + (Y_i - \alpha_i)^2 - 2(Y_i - \alpha_i)\theta + \theta^2 \right\}$$

Maximizing the likelihood in α_i, θ is equivalent to

minimizing the quadratic $\Sigma (X_i - \alpha_i)^2 + (Y_i - \alpha_i)^2 - 2(Y_i - \alpha_i)\theta + \theta^2$.

let $Q(\alpha_i, \theta) = \Sigma\left[(X_i - \alpha_i)^2 + (Y_i - \alpha_i)^2 - 2(Y_i - \alpha_i)\theta + \theta^2 \right]$

$$\therefore \frac{\partial Q}{\partial \alpha_i} = -2(X_i - \alpha_i) - 2(Y_i - \alpha_i) + 2\theta$$

$$\therefore \frac{\partial Q}{\partial \alpha_i^2} = +4 \qquad \frac{\partial Q}{\partial \alpha_i \partial \alpha_j} = 0 \quad \forall i \neq j$$

$$\frac{\partial Q}{\partial \theta} = \Sigma\left[-2(Y_i - \alpha_i) + 2\theta \right] \quad \text{Ⅲ}$$

$$\frac{\partial Q}{\partial \theta^2} = 2n \qquad \frac{\partial Q}{\partial \theta \partial \alpha_i} = 2$$

$\therefore$ the Hessian matrix is

$$H = \begin{pmatrix} 2n & 2 & 2 & \cdots & 2 \\ 2 & 4 & 0 & \cdots & 0 \\ 2 & 0 & 4 & & \vdots \\ \vdots & \vdots & & \ddots & \\ 2 & 0 & — & & 4 \end{pmatrix}$$

$$\therefore \quad v^T H v = (v_1, \ldots, v_n) \begin{pmatrix} 2nv_1 + 2v_2 + \cdots + 2v_n \\ 2v_1 + 4v_2 \\ 2v_1 + 4v_3 \\ \vdots \\ 2v_1 + 4v_n \end{pmatrix}$$

$$= \Big\{ \cancel{2nv_1^2 + 2v_1v_2 + \cdots + 2v_1v_n}$$
$$= 2nv_1^2 + 2v_1v_2 + \cdots + 2v_1v_n + 2v_1v_2 + 4v_2^2 + 2v_1v_3 + 4v_3^2 + \cdots$$

$= 2n v_1^2 + 4v_2^2 + \dots + 4v_n^2 + 4v_1v_2 + \dots + 4v_1v_n$

$= 2(v_1+v_2)^2 + 2(v_1+v_3)^2 + \dots + 2(v_1+v_n)^2 + 2v_2^2 + 2v_3^2 + \dots + 2v_n^2$

≥ 0 $\therefore$ H is +ve definite

$\therefore$ Our stationary point $\frac{\partial Q}{\partial \alpha_i} = \frac{\partial Q}{\partial \theta} = 0$ is the MLE

$\therefore$ $\hat{\alpha}_i = \frac{x_i + y_i - \hat{\theta}}{2}$, $\hat{\theta} \overset{Ⓘⓘ}{=} \frac{\Sigma y_i - \hat{\alpha}_i}{n}$

~~$\therefore \hat{\alpha}_i = x_i + y_i$~~ Summing the left equation in i gives

$\Sigma \hat{\alpha}_i = \frac{\Sigma x_i + y_i}{2} + \frac{n}{2}\hat{\theta} \xleftarrow{\Sigma}$ plugging into II,

$\therefore$ $\hat{\theta} = \frac{(\Sigma y_i) - \left(\frac{\Sigma x_i + y_i}{2} + \frac{n}{2}\hat{\theta}\right)}{n}$

$= \frac{\Sigma y_i - x_i}{2n} - \frac{1}{2}\hat{\theta}$

$\therefore$ $\hat{\theta}_{MLE} = \frac{\Sigma y_i - x_i}{\cancel{2}n} = \bar{Y} - \bar{X} \sim N(\theta, \frac{2\sigma^2}{n})$

(and $\hat{\alpha}_i^{MLE} = \frac{x_i + y_i - (\bar{Y} - \bar{X})}{2}$)

From III, $\frac{\partial^2 l}{\partial \theta^2} = -\frac{1}{2\sigma^2}\left[\Sigma\, 2n\right] = -\frac{n}{\sigma^2}$

$\therefore$ $E - \frac{\partial^2 l}{\partial \theta^2} = \frac{n}{\sigma^2}$, which has MLE $\frac{n}{\sigma^2}$ where

σ^2 maximizes $l(\hat{\alpha}_i^{MLE}, \hat{\theta}^{MLE}, \hat{\sigma}^2; X, Y)$

2001 Q3

It is easy to check that this is

$$\hat{\sigma}^2 = \frac{1}{2n}\left[\sum (X_i-\hat{\alpha}_i)^2 + (Y_i-\hat{\alpha}_i)^2 - 2(Y_i-\hat{\alpha}_i)\hat{\theta} + \hat{\theta}^2\right]$$

$$= \frac{1}{2n}\left[\sum\left(\frac{X_i-Y_i}{2} + \frac{\bar{Y}-\bar{X}}{2}\right)^2 + \left(\frac{Y_i-X_i}{2} + \frac{\bar{Y}-\bar{X}}{2}\right)^2 - 2\left(\frac{Y_i-X_i}{2} + \frac{\bar{Y}-\bar{X}}{2}\right)(\bar{Y}-\bar{X}) + (\bar{Y}-\bar{X})^2\right]$$

$$= \frac{1}{2n}\left[\sum 2\frac{(Y_i-X_i)^2}{4} + 2\frac{(\bar{Y}-\bar{X})^2}{4} - (Y_i-X_i)(\bar{Y}-\bar{X}) \cancel{+} - (\bar{Y}-\bar{X})^2 + (\bar{Y}-\bar{X})^2\right]$$

$$= \frac{1}{2n}\left[\frac{1}{2}\sum (Y_i-X_i)^2 - n(\bar{Y}-\bar{X})^2 + \frac{1}{2}n(\bar{Y}-\bar{X})^2\right]$$

$$= \frac{1}{4n}\sum (Y_i-X_i)^2 \cancel{+} - \cancel{\tfrac{3}{2}}\tfrac{1}{4}(\bar{Y}-\bar{X})^2 \xrightarrow[\text{WLLN}]{P} \cancel{\frac{\theta^2}{2} + \sigma^2 + \frac{3}{2}\theta^2 = 2\theta^2+\sigma^2}$$

$$\frac{1}{4}(\theta^2+2\sigma^2) - \frac{1}{4}\theta^2 = \frac{1}{2}\sigma^2$$

$$\therefore \hat{I} = \frac{n}{\hat{\sigma}^2} = \frac{n}{\frac{\sum (Y_i-X_i)^2}{4n} - \cancel{\tfrac{3}{2}}\tfrac{1}{4}(\bar{Y}-\bar{X})^2}$$

$$\therefore \frac{\hat{\theta}-\theta}{\sqrt{\hat{I}}} = \frac{(\bar{Y}-\bar{X}-\theta)\sqrt{\frac{\sum (Y_i-X_i)^2}{4n} - \cancel{\tfrac{3}{2}}\tfrac{1}{4}(\bar{Y}-\bar{X})^2}}{\sqrt{n}} \xrightarrow{P} 0 \quad \text{(Slutsky's)}$$

$$(\hat{\theta}-\theta)\sqrt{\hat{I}} = \frac{\sqrt{n}(\bar{Y}-\bar{X}-\theta)}{\sqrt{\frac{\sum (Y_i-X_i)^2}{4n} + \cancel{\tfrac{3}{2}}\tfrac{1}{4}(\bar{Y}-\bar{X})^2}} \xrightarrow{d} \frac{N(0, 2\sigma^2)}{\sqrt{\cancel{2\theta^2 +} \frac{\sigma^2}{2}}} = \cancel{N\left(0, \frac{2\sigma^2}{2\theta^2+\sigma^2}\right)} \quad \boxed{N(0,4)}$$

by Slutsky's

Cannot apply standard MLE theory as we do not have iid observations.

2001 Q6

Fix $\theta_0 \in \mathbb{R}$.

As g is well behaved,

$$\frac{\partial}{\partial\theta} E_\theta g(X) = \frac{\partial}{\partial\theta} \int g(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2} dx$$

$$= \int g(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2} \left(+\tfrac{1}{2}\cdot 2(x-\theta)\right) dx$$ (by class results on $N(\theta,1)$ is exp. fam.)

$$= \int g(x)(x-\theta) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2} dx$$

$$= E_\theta\left[g(X)(X-\theta)\right]$$

$\therefore$ $\boxed{h(X) = X - \theta_0}$ satisfies the desired property

Extra: $\frac{\partial}{\partial\theta} E_\theta g(X) = \frac{\partial}{\partial\theta} \int g(x) f_\theta(x)\,dx$

$$= \int g(x) \frac{\partial}{\partial\theta} f_\theta(x)\,dx$$ (g well behaved, f_θ smooth)

$$= \int g(x) \frac{\frac{\partial}{\partial\theta} f_\theta(x)}{f_\theta(x)} f_\theta(x)\,dx$$

$$= \cancel{2}\, E_\theta\left[g(X) \frac{\partial}{\partial\theta} f_\theta(X) \cdot \frac{1}{f_\theta(X)}\right]$$

$$\therefore \boxed{h(X) = \cancel{2}\, \frac{1}{f_{\theta_0}(X)} \cdot \frac{\partial}{\partial\theta} f_\theta(X)\Big|_{\theta=\theta_0}}$$ $\square$

~~$(\sigma_2 \; \rho Z_1 + \sqrt{1-\rho^2}\, Z_2))$~~

2001 Q7

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\right)$$

$$\therefore \quad \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \sim N\left(0, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\right)$$

$Z \perp\!\!\!\perp X_1$

$\boxed{X_2 = \rho X_1 + \sqrt{1-\rho^2}\, Z}$

let $X_1 = \frac{Y_1 - \mu_1}{\sigma_1}$ $\quad X_2 = \frac{Y_2 - \mu_2}{\sigma_2}$

then $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$ where $\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$.

let $\tilde{Z}_1 = X_1$, $\tilde{Z}_2 = X_2 - \rho X_1$

then $\begin{pmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\rho & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

since $\begin{pmatrix} 1 & 0 \\ -\rho & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} 1 & -\rho \\ 0 & 1 \end{pmatrix}}$ =

$$= \begin{pmatrix} 1 & 0 \\ -\rho & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \rho & 1-\rho^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-\rho^2 \end{pmatrix}$$

letting $Z_1 = \tilde{Z}_1$, $Z_2 = \tilde{Z}_2 / \sqrt{1-\rho^2}$, we find

$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N(0, I_2)$

Thus

$$E\, Y_1^2 Y_2^2 = E\left[(\mu_1 + \sigma_1 X_1)^2 (\mu_2 + \sigma_2 X_2)^2\right]$$

$$= E\left[(\mu_1 + \sigma_1 \tilde{Z}_1)^2 (\mu_2 + \sigma_2 (\tilde{Z}_2 + \rho \tilde{Z}_1))^2\right]$$

$$= E\left[(\mu_1+\sigma_1 z_1)^2 \left(\mu_2+\sigma_2\left(z_1\sqrt{1-\rho^2} + \rho z_1\right)\right)^2\right]$$

$$= E\left[\left(\mu_1^2 + 2\mu_1\sigma_1 z_1 + \sigma_1^2 z_1^2\right)\left(\mu_2^2 + 2\mu_2\sigma_2\left(z_2\sqrt{1-\rho^2} + \rho z_1\right) + \sigma_2^2\left(z_2\sqrt{1-\rho^2} + \rho z_1\right)^2\right)\right]$$

$$= \mu_1^2\mu_2^2 + \sigma_1^2\mu_2^2 + 4\mu_1\mu_2\sigma_1\sigma_2\rho + 2\mu_2\sigma_2\sigma_1^2\rho\, E z_1^3 + \sigma_1^2\sigma_2^2\, E\left[z_1^2\left((1-\rho^2)z_2^2 + 2\rho\sqrt{1-\rho^2}\, z_1 z_2 + \rho^2 z_1^2\right)\right]$$
$$+ \mu_1^2\sigma_2^2\, E\left(z_2\sqrt{1-\rho^2} + \rho z_1\right)^2 + 2\mu_1\sigma_1\sigma_2^2\, E\left[z_1\left(z_2\sqrt{1-\rho^2} + \rho z_1\right)^2\right]$$

$$= \mu_1^2\mu_2^2 + \sigma_1^2\mu_2^2 + 4\mu_1\mu_2\sigma_{12} + 2\mu_2\sigma_1^2\sigma_2\rho\, E z_1^3 + \sigma_1^2\sigma_2^2(1-\rho^2) + \sigma_1^2\sigma_2^2\rho^2\, E z_1^4$$
$$+ \mu_1^2\sigma_2^2(1-\rho^2) + \mu_1^2\sigma_2^2\rho^2 + 0$$

$$= \mu_1^2\mu_2^2 + \sigma_1^2\mu_2^2 + 4\mu_1\mu_2\sigma_{12} + \sigma_1^2\sigma_2^2\left(1 - \frac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2}\right) + \sigma_{12}^2\left\{\mathrm{Var}\,\chi_1^2 + E^2\chi_1^2\right\} + \mu_1^2\sigma_2^2$$

$$= \mu_1^2\mu_2^2 + \sigma_1^2\mu_2^2 + 4\mu_1\mu_2\sigma_{12} + \left(\sigma_1^2\sigma_2^2 - \sigma_{12}^2\right) + 3\sigma_{12}^2 + \mu_1^2\sigma_2^2$$

$$\boxed{= \mu_1^2\mu_2^2 + \sigma_1^2\mu_2^2 + \sigma_2^2\mu_1^2 + 4\mu_1\mu_2\sigma_{12} + \sigma_1^2\sigma_2^2 + 2\sigma_{12}^2 .}$$

2001 Q7

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\underbrace{\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}}_{\mu}, \underbrace{\begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}}_{\Sigma} \right)$$

$$\therefore \quad Y_1 + Y_2 = (1 \ 1)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left((1 \ 1)\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, (1 \ 1)\begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$
$$= N(\mu_1 + \mu_2, \ \sigma_1^2 + \sigma_{12} + \sigma_{12} + \sigma_2^2)$$

~~Let $X_1 = \frac{Y_1 - \mu_1}{\sigma_1}$ $X_2 = \frac{Y_2 - \mu_2}{\sigma_2}$~~

~~$\therefore \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1/\sigma_1 \\ \mu_2/\sigma_2 \end{pmatrix}, \begin{pmatrix} 1 & \frac{\sigma_{12}}{\sigma_1\sigma_2} \\ \frac{\sigma_{12}}{\sigma_1\sigma_2} & 1 \end{pmatrix} \right)$~~

~~Let $\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$~~

Likewise, $Y_1 - Y_2 = (1 \ -1)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N(\mu_1 - \mu_2, \ \sigma_1^2 - 2\sigma_{12} + \sigma_2^2)$

Thus, $E(Y_1+Y_2)^4 = EY_1^4 + 4EY_1Y_2^3 + 6EY_1^2Y_2^2 + 4EY_1^3Y_2 + EY_2^4$

$E(Y_1 - Y_2)^4 = EY_1^4 - 4EY_1^3Y_2 + 6EY_1^2Y_2^2 - 4EY_1Y_2^3 + EY_2^4$

Now note $E\,N(\mu, \sigma^2)^4 = E(\mu + \sigma N(0,1))^4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$

$$\therefore \quad E(Y_1+Y_2)^4 + E(Y_1-Y_2)^4 = (\mu_1+\mu_2)^4 + 6(\mu_1+\mu_2)^2(\sigma_1^2 + 2\sigma_{12} + \sigma_2^2) + 3(\sigma_1^2 + 2\sigma_{12} + \sigma_2^2)^2$$
$$+ (\mu_1 - \mu_2)^4 + 6(\mu_1 - \mu_2)^2(\sigma_1^2 - 2\sigma_{12} + \sigma_2^2) + 3(\sigma_1^2 - 2\sigma_{12} + \sigma_2^2)$$

$$\therefore \quad 2EY_1^4 + 12EY_1^2Y_2^2 + 2EY_2^4 =$$
$$= 2\mu_1^4 + 2\mu_2^4 + 12\mu_1^2\mu_2^2 + 12(\mu_1^2 + \mu_2^2)(\sigma_1^2 + \sigma_2^2) + 48\mu_1\mu_2\sigma_{12} + 6\sigma_1^4 + 6\sigma_2^4 + 12\sigma_1^2\sigma_2^2$$
$$+ 24\sigma_{12}^2$$

$$\therefore \quad 12EY_1^2Y_2^2 = 12\mu_1^2\mu_2^2 + 12\mu_1^2\sigma_2^2 + 12\mu_2^2\sigma_1^2 + 48\mu_1\mu_2\sigma_{12} + 12\sigma_1^2\sigma_2^2 + 24\sigma_{12}^2$$

$$\boxed{\therefore \quad EY_1^2Y_2^2 = \mu_1^2\mu_2^2 + \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + 4\mu_1\mu_2\sigma_{12} + \sigma_1^2\sigma_2^2 + 2\sigma_{12}^2}$$

2000 Q2

The problem is equivalent to $(X_i, Y_i)^T \overset{iid}{\sim} N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$

where $\mu_1^2 + \mu_2^2 = 1$, i.e. the means lie on the unit circle.

$\therefore$ the MLE's $(\hat\mu_1, \hat\mu_2)$ solve

$$(\hat\mu_1, \hat\mu_2) = \underset{\mu_1^2+\mu_2^2=1}{\operatorname{argmax}} \; l(\mu_1, \mu_2; \vec{x}, \vec{y})$$

Now $l(\mu_1, \mu_2; \vec{x}, \vec{y}) = \text{constant} - \frac{1}{2}\sum_1^n (x_i - \mu_1)^2 - \frac{1}{2}\sum_1^n (y_i - \mu_2)^2$

$$= \text{constant} - \frac{n}{2}\cancel{\Sigma}(\bar{x} - \mu_1)^2 - \frac{n}{2}\cancel{\Sigma}(\bar{y} - \mu_2)^2 .$$

To minimize l subject to $g(\vec{\mu}) = \mu_1^2 + \mu_2^2 = 1$, consider

$$\mathcal{L}(\mu_1, \mu_2, \lambda) = l + \lambda g$$

$$= -\frac{n}{2}\cancel{\Sigma}(\bar{x} - \mu_1)^2 - \frac{n}{2}\cancel{\Sigma}(\bar{y} - \mu_2)^2 + \lambda(\mu_1^2 + \mu_2^2)$$

$$= -\frac{n}{2}\bar{x}^2 + n\bar{x}\mu_1 - \frac{n}{2}\mu_1^2 - \frac{n}{2}\bar{y}^2 + n\bar{y}\mu_2 - \frac{n}{2}\mu_2^2 + \lambda\mu_1^2 + \lambda\mu_2^2$$

Maximizing $\mathcal{L}$ in μ_1, μ_2 gives

$$\hat\mu_1(\lambda) = \frac{n\bar{x}}{n - \frac{\lambda}{2}} \qquad \hat\mu_2(\lambda) = \frac{n\bar{y}}{n - \frac{\lambda}{2}}$$

provided $\lambda < \cancel{\frac{n}{2}}\, 2n$
(otherwise no maximiser ~~minimiser~~ exists)

as we have quadratics in these two variables.

imposing our constraint $g(\hat\mu_1(\lambda), \hat\mu_2(\lambda)) = 1$ gives

$$n^2\bar{x}^2 + n^2\bar{y}^2 = \left(n - \frac{\lambda}{2}\right)^2$$

$$\therefore \quad n^2\bar{x}^2 + n^2\bar{y}^2 = n^2 - n\lambda + \frac{\lambda^2}{4}$$

$$\therefore \quad \frac{1}{4}\lambda^2 - n\lambda + n^2 - n^2\bar{x}^2 - n^2\bar{y}^2 = 0$$

$$\therefore \quad \lambda = \frac{n \pm \sqrt{n^2 - (n^2 - n^2\bar{x}^2 - n^2\bar{y}^2)}}{1/2}$$

$$= 2n \pm 2n\sqrt{\bar{x}^2 + \bar{y}^2}$$

$$= 2n\left(1 \pm \sqrt{\bar{x}^2 + \bar{y}^2}\right)$$

∴ But ~~a~~ we required $\lambda < 2n$ in order for there to exist a ~~global~~ maximizer. ∴ ~~$\hat{\lambda} = 2n - \sqrt{}$~~

$$\therefore \quad \hat{\lambda} = 2n\left(1 - \sqrt{\bar{x}^2 + \bar{y}^2}\right)$$

$$\boxed{\therefore \quad (\hat{\mu}_1, \hat{\mu}_2) = \left(\frac{\bar{x}}{\sqrt{\bar{x}^2+\bar{y}^2}}, \frac{\bar{y}}{\sqrt{\bar{x}^2+\bar{y}^2}}\right)}$$ solves our optimization and is the MLE.

(b) At $\theta = 0$, $\mu_1 = 0$, $\mu_2 = 1$

Also, by CLT, $\sqrt{n}\left(\begin{pmatrix}\bar{X}\\ \bar{Y}\end{pmatrix} - \begin{pmatrix}0\\ 1\end{pmatrix}\right) \xrightarrow{d} N(\vec{0}, I_2)$

Let ~~g~~ $h(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

then $$\frac{\partial h_1}{\partial x} = \frac{\sqrt{x^2+y^2} - x\left(\frac{1}{2}\cdot\frac{1}{\sqrt{x^2+y^2}}\cdot(2x)\right)}{x^2+y^2} = \frac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial h_1}{\partial y} = \frac{-xy}{\left(\sqrt{x^2+y^2}\right)^3} = \frac{-xy}{(x^2+y^2)^{3/2}}$$

2000 Q2

And similarly,

$$\frac{\partial h_2}{\partial x} = \frac{-xy}{(x^2+y^2)^{3/2}} \qquad \frac{\partial h_2}{\partial y} = \frac{x^2}{(\sqrt{x^2+y^2})^3}$$

$$\therefore \quad \frac{\partial \vec{h}}{\partial(x,y)} = \begin{pmatrix} \frac{y^2}{(x^2+y^2)^{3/2}} & \frac{-xy}{(x^2+y^2)^{3/2}} \\ \frac{-xy}{(x^2+y^2)^{3/2}} & \frac{x^2}{(x^2+y^2)^{3/2}} \end{pmatrix}$$

at ~~β~~ $(x,y) = (0,1)$ this becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

By the multivariate Δ-theorem,

$$\boxed{\begin{aligned} \sqrt{n}\left(\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right) &\xrightarrow{d} N\left(\vec{0},\ H^T I_2 H\right) \\ &= N\left(\vec{0},\ \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right) \end{aligned}}$$

2000 Q5

b $p_\lambda(x) = \dfrac{e^{-n\lambda}\lambda^{\Sigma x_i}}{\prod x_i!} = \exp\{(\Sigma x_i)(\log\lambda) - n\lambda\} \prod_1^n x_i!^{-1}$

This is an exponential family with natural parameter

$\eta(\lambda) = \log\lambda$. As $\lambda \in (0,\infty)$, $\tilde{\eta} = \{\eta(\lambda) : \lambda \in (0,\infty)\}$

has non-empty interior, therefore $T(X) = \sum_1^n X_i$ is M.S and C.S.

For the UMVUE, note that

$$E\,T^2 = \sum_1^n E X_i^2 + \sum_{i \neq j} E X_i X_j$$

$$= n(\lambda^2 + \lambda) + n(n-1)\lambda^2 \qquad (E X_1^2 = \operatorname{Var} X_1 + E^2 X_1 = \lambda - \lambda^2)$$

$$= n^2\lambda^2 + n\lambda$$

$\therefore \quad \frac{1}{n^2} T^2 - \frac{1}{n^2} T$ is UMVUE (Rao-Blackwell thm)

i.e. $\boxed{\bar{X}^2 - \frac{1}{n}\bar{X}}$

2000 Q7

(a) $E\hat{\theta} = \frac{1}{mn} \sum_i \sum_j E f(X_i, X_j)$

$= \frac{1}{mn} \sum_i \sum_j \iint f(x,y)\, dx\, dy$

$= \theta \qquad \square$

(b) We first calculate the quantity we are aiming to estimate:

$$\text{Var}\,\hat{\theta} = \frac{1}{m^2n^2} \text{Var} \sum_{i,j} f(X_i, Y_j)$$

$$= \frac{1}{m^2n^2}\left[\sum_{i,j} \text{Var}\, f(X_i,Y_j) + \sum_{(i,j)\neq(l,k)} \text{Cov}\left(f(X_i,Y_j), f(X_l,Y_k)\right)\right]$$

$$= \frac{1}{m^2n^2}\left[mn\,\text{Var}\, f(X_1,Y_1) + \sum_{\substack{i,j,l,k:\\ i=l,\, j\neq k}} \text{Cov}\left(f(X_i,Y_j), f(X_l,Y_k)\right)\right.$$

$$\left. + \sum_{\substack{i,j,l,k\\ i\neq l,\, j=k}} \text{Cov}\left(f(X_i,Y_j), f(X_l,Y_k)\right) + \sum_{\substack{i,j,l,k\\ i\neq l,\, j\neq k}} \text{Cov}\left(f(X_i,Y_j), f(X_l,Y_k)\right)\right]$$

$$= \frac{1}{m^2n^2}\left[mn\,\text{Var}\, f(X_1,Y_1) + nm(m-1)\,\text{Cov}\left(f(X_1,Y_2), f(X_1,Y_2)\right) + mn(m-1)(n-1)\,\text{Cov}\left(f(X_1,Y_1), f\right.\right.$$

$$+ mn(n-1)\,\text{Cov}\left(f(X_1,Y_1), f(X_2,Y_1)\right)$$

Now compute

$$\text{Var}\, f(X_1,Y_1) = E f(X_1,Y_1)^2 - E^2 f(X_1,Y_1) = \iint f^2(x,y)\, dx\, dy - \theta^2$$

$$\text{Cov}\left(f(X_1,Y_1), f(X_1,Y_2)\right) = E f(X_1,Y_1) f(X_1,Y_2) - E f(X_1,Y_1) E f(X_1,Y_2)$$

$$= \iiint f(x,y) f(x,\tilde{y})\, dx\, dy\, d\tilde{y} - \theta^2 .$$

~~$\left(\mathrm{Cov}\left(f(X_1,Y_1), f(X_2,Y_2)\right) = \iiint f(x,y) f(\tilde{x},y)\, dx\, d\tilde{x}\, dy - \theta^2 \right)$~~

$\mathrm{Cov}\left(f(X_1,Y_1), f(X_2,Y_2)\right) = 0$ by independence.

$$\therefore \mathrm{Var}\ \hat{\theta} = \frac{1}{m^2 n^2}\left[mn \iint f(x,y)^2\, dx\, dy + nm(m-1) \iiint f(x,y) f(x,\tilde{y})\, dx\, dy\, d\tilde{y} \right.$$
$$\left. + mn(n-1) \iiint f(x,y) f(\tilde{x},y)\, dx\, d\tilde{x}\, dy - \left(mn + nm(m-1) + mn(n-1)\right)\theta^2 \right]$$

$\therefore$ It suffices to provide unbiased estimators of $\iint f(x,y)^2\, dx\, dy$, $\iiint f(x,y) f(x,\tilde{y})\, dx\, dy\, d\tilde{y}$, $\iiint f(x,y) f(\tilde{x},y)\, dx\, d\tilde{x}\, dy$ and θ^2, and use linearity.

To ~~conclude, note E[f~~

Now simply note that

~~$E[f(x_1,y_1)f(x_2,y_2)$~~

$$E\left[f(x_1,y_1)f(x_1,y_2) - f(x_1,y_1)f(x_2,y_2)\right] = \mathrm{Cov}\left(f(X_1,Y_1), f(X_1,Y_2)\right)$$
$$E\left[f(x_1,y_1)f(x_2,y_1) - f(x_1,y_1)f(x_2,y_2)\right] = \mathrm{Cov}\left(f(X_1,Y_1), f(X_2,Y_1)\right)$$
$$E\left[\tfrac{1}{2}\left(f(x_1,y_1) - f(x_2,y_2)\right)^2\right] = \tfrac{1}{2} E f(X_1,Y_1)^2 - \cancel{2} E f(X_1,Y_1) E f(X_2,Y_2) + \tfrac{1}{2} E f(X_2,Y_2)^2$$
$$= E f(X_1,Y_1)^2 - E^2 f(X_1,Y_1) = \mathrm{Var}\ f(X_1,Y_1).$$

Thus, an unbiased estimator of $\mathrm{Var}\ \hat{\theta}$, by linearity, is

$$T = \frac{1}{m^2 n^2}\left[mn \left\{ \tfrac{1}{2}\left(f(x_1,y_1) - f(x_2,y_2)\right)^2 \right\} + nm(m-1) \left\{ f(x_1,y_1) f(x_1,y_2) - f(x_1,y_1) f(x_2,y_2) \right\} \right.$$
$$\left. + mn(n-1) \left\{ f(x_1,y_1) f(x_2,y_1) - f(x_1,y_1) f(x_2,y_2) \right\} \right] \qquad \square$$

1999 Q2

(a) Let $E Y_i | X_i = \alpha + \beta X_i$, $x_i \in \{0,1\}$ and we aim to minimise the MSE for estimating β.

Here, $\begin{pmatrix}\hat\alpha \\ \hat\beta\end{pmatrix} = (X^T X)^{-1} X^T Y$

where $X = \begin{pmatrix}1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1\end{pmatrix}$ $\begin{matrix}\} \; n_0 \text{ observations} \\ \} \; n_1 \text{ observations}\end{matrix}$ $\quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_{n_0} \\ y_{n_0+1} \\ \vdots \\ y_{n_0+n_1}\end{pmatrix}$

$\therefore (X^T X) = \begin{pmatrix} n_0+n_1 & n_1 \\ n_1 & n_1 \end{pmatrix} \quad \therefore (X^T X)^{-1} = \frac{1}{(n_0+n_1)n_1 - n_1^2}\begin{pmatrix} n_1 & -n_1 \\ -n_1 & n_0+n_1\end{pmatrix}$

$X^T Y = \begin{pmatrix} \sum_1^{100} Y_i \\ \sum_{n_0+1}^{100} Y_i \end{pmatrix}$

$\therefore (X^T X)^{-1} X^T Y = \frac{1}{n_0 n_1}\begin{pmatrix} n_1 \sum_1^{n_0} Y_i \\ n_0 \sum_{n_0+1}^{100} Y_i - n_1 \sum_1^{n_0} Y_i \end{pmatrix}$

$\therefore \hat\beta = \frac{1}{n_1}\sum_{n_0+1}^{100} Y_i - \frac{1}{n_0}\sum_1^{n_0} Y_i$

$\therefore E\hat\beta = \beta$, $\operatorname{Var}\hat\beta = \frac{1}{n_1}\sigma^2 + \frac{1}{n_0}\sigma^2 = \frac{100}{n_1 n_0}\sigma^2 = \frac{100}{n_0(100-n_0)}\sigma^2$

minimise MSE $\iff$ maximise $n_0(100-n_0) = -n_0^2 + 100 n_0$

$= -(n_0-50)^2 + 2500$

$\iff n_0 = n_1 = 50.$ □

(b) Our estimate of $g(2)$ is $\hat\alpha + 2\hat\beta$.

Writing $\bar{Y}_0 = \frac{1}{n_0}\sum_1^{n_0} Y_i$, $\bar{Y}_1 = \frac{1}{n_1}\sum_{n_0+1}^{100} Y_i$, we find that

$\cancel{E\,\hat{\alpha}+2}\quad \hat{\alpha}+2\hat{\beta} = \bar{Y}_0 + 2(\bar{Y}_1 - \bar{Y}_0) = 2\bar{Y}_1 - \bar{Y}_0$

$\therefore\quad E\,\hat{\alpha}+2\hat{\beta} = \alpha + 2\beta = g(2)$

$$\mathrm{Var}(\hat{\alpha}+2\hat{\beta}) = 4\,\mathrm{Var}\,\bar{Y}_1 + \mathrm{Var}\,\bar{Y}_0$$

$$= 4\,\frac{1}{n_1}\sigma^2 + \frac{1}{n_0}\sigma^2$$

$$= \frac{4n_0 + n_1}{n_0 n_1}\sigma^2$$

$$= \frac{100 + 3n_0}{n_0(100-n_0)}\sigma^2$$

to minimise this we compute

$$\frac{\partial}{\partial n}\left(\frac{100+3n}{100n-n^2}\right) = \frac{(100n-n^2)3 - (100+3n)(100-2n)}{(100n-n^2)^2} = 0$$

$$\Rightarrow \cancel{-3n^2}\quad 300n - 3n^2 - 10000 - 100n + 6n^2 = 0$$

$$\Rightarrow +3n^2 + 200n - 10000 = 0$$

$$\Rightarrow n = \frac{-200 \pm \sqrt{40000 + 4\,3\cdot 10000}}{\cancel{8}\ 6} = \frac{-200 \pm \sqrt{160000}}{6} = \frac{-200 \pm 400}{6} = \frac{200}{6}\ \boxed{= \frac{100}{3}}$$

~~$\therefore\ n_0 \approx 16.6,\quad n_1 \approx 83.4$ is optimal~~

$$\left(\frac{\partial^2}{\partial n^2}\left(\frac{100+3n}{100n-n^2}\right) = \frac{(100n-n^2)^2(-6n+400) - (-3n^2+400n-10000)(2(100n-n^2)(100-2n))}{(100n-n^2)^4} < 0\right.$$

$$\left. = \cancel{\left(\frac{10000\cdot 8}{9}\right)^2 (300)}\right)$$

$\therefore\quad n_0 \approx 33.3,\quad n_1 \approx 66.6$ is optimal $\square$

1999 Q5

(a) MLE maximises $L(\theta; X_1) = \dfrac{1}{\pi(1+(X_1-\theta)^2)}$

i.e. minimizes $1+(X_1-\theta)^2$

$\therefore \hat{\theta} = X_1$

(b) MLE maximises $L(\theta; X_1, X_2) = \dfrac{1}{\pi^2(1+(X_1-\theta)^2)(1+(X_2-\theta)^2)}$

i.e. minimizes $(1+(x_1-\theta)^2)(1+(X_2-\theta)^2) =$

$$= 1+(X_1-\theta)^2+(X_2-\theta)^2+(X_1-\theta)^2(X_2-\theta)^2 \qquad (I)$$

Derivative is:

$$(II) \qquad -2(X_1-\theta)-2(X_2-\theta)-2(X_1-\theta)(X_2-\theta)^2-2(X_1-\theta)^2(X_2-\theta) \; = 0$$

This has a roots: $\theta = \dfrac{X_1+X_2}{2}$

~~X_2~~

~~$\dfrac{X_1+X_2}{2}$~~

2nd derivative is:

$$(III) \qquad 2+2+2(X_2-\theta)^2+2(X_1-\theta)^2+8(X_1-\theta)(X_2-\theta)$$

At $\theta = \dfrac{X_1+X_2}{2}$ this is $4+\dfrac{(X_1-X_2)^2}{[illegible]}-2(X_1-X_2)^2$

$$= 4 - [illegible](X_1-X_2)^2 \qquad (IV)$$

Rewrite III as

$$\text{III} = 4\left(\theta - \frac{x_1+x_2}{2}\right) - 2(x_1-\theta)(x_2-\theta)\overbrace{\left(x_1-\theta+x_2-\theta\right)}^{-2\left(\theta-\frac{x_1+x_2}{2}\right)}$$

$$= \left[4 + 4(x_1-\theta)(x_2-\theta)\right]\left(\theta - \frac{x_1+x_2}{2}\right)$$

Note $1 + (\theta-x_1)(\theta-x_2) = 0 \implies$

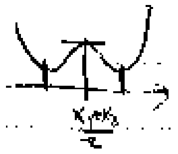
$$\implies \theta^2 - (x_1+x_2)\theta + x_1x_2 + 1 = 0$$

$$\implies \theta = \frac{x_1+x_2 \pm \sqrt{(x_1+x_2)^2 - 4(x_1x_2+1)}}{2} = \frac{x_1+x_2 \pm \sqrt{(x_1-x_2)^2 - 4}}{2}$$

$\therefore$ We have 2 cases.

Case 1 if $(x_1-x_2)^2 \le 4$, i.e. $|x_1-x_2| \le 2$,
then $\exists$ unique root $\hat{\theta} = \bar{x}$ and by IV it is a minimum $\square$

Case 2: if ~~$(x_1-x_2)^2 > 16$~~ $|x_1-x_2| > 2$, then $\exists$ 3 roots
and by IV they ~~took~~ look like this



As, by symmetry, the function I
evaluates to the same result at $\theta+\delta$ and $\theta-\delta$,
it follows that

$$\hat{\theta}_{MLE} = \frac{x_1+x_2 \pm \sqrt{(x_1-x_2)^2 - 4}}{2} = \frac{x_1+x_2}{2} \pm \sqrt{\frac{(x_1-x_2)^2}{2} - 1}$$

with both roots being optimal in this case.

1999 Q7

(a) If θ is known, the likelihood is

$$L(\mu;X) = \prod_{i}^{n} \frac{1}{\theta} e^{-\frac{x_i-\mu}{\theta}} \mathbb{1}_{\{x_i \geq \mu\}}$$

$$= \underbrace{\theta^{-n} e^{-\frac{\sum x_i}{\theta}}}_{\text{function of } \vec{X}} \underbrace{e^{\frac{n}{\theta}\mu} \mathbb{1}_{\{x_{(1)} \geq \mu\}}}_{\text{function of } (\mu, X_{(1)})}$$

By Neyman Fisher factorisation criterion, $X_{(1)}$ is sufficient for μ ☐

(b) In this case,

$$L(\mu,\theta;X) = \theta^{-n} \mathbb{1}_{\{X_{(1)} \geq \mu\}} \exp\left\{ \frac{n\mu}{\theta} - \frac{n\bar{X}}{\theta} \right\}$$

Clearly, $\forall \theta \in \mathbb{R}^+$, $\exp\left\{\frac{n\mu}{\theta}\right\}$ is increasing in μ,

while $\mathbb{1}_{\{X_{(1)} \geq \mu\}} = \begin{cases} 1 & \text{if } \mu \leq X_{(1)} \\ 0 & \text{o/w} \end{cases}$

Therefore, $\hat{\mu} = X_{(1)}$ clearly maximises $L(\mu,\theta;X)$ regardless of the value of θ. Therefore, to find $\hat{\theta}$, it suffices to maximise $L(\hat{\mu},\theta;X) = \theta^{-n} \exp\left\{ \frac{nX_{(1)}}{\theta} - \frac{n\bar{X}}{\theta} \right\}$

taking logs,

$\Rightarrow \ell(\hat{\mu},\theta;X) = -n\log\theta + \frac{n(X_{(1)} - \bar{X})}{\theta}$

$\therefore \ell' \frac{\partial \ell}{\partial \theta}(\hat{\mu},\theta;X) = -\frac{n}{\theta} + \frac{n(\bar{X} - X_{(1)})}{\theta^2}$, which has a unique root at $\hat{\theta} = \bar{X} - X_{(1)}$.

$\therefore \frac{\partial^2 \ell}{\partial \theta^2} = + \frac{n}{\theta^2} - 2\frac{n(\bar{X} - X_{(1)})}{\theta^3}$, which is negative at $\hat{\theta}$, since

$$\frac{\partial^2 l}{\partial\theta^2}(\hat{\mu},\hat{\theta};X) = \frac{n}{(\bar{X}-X_{(1)})^2} - 2\,\frac{n(\bar{X}-X_{(1)})}{(\bar{X}-X_{(1)})^3} = -\frac{n}{(\bar{X}-X_{(1)})^2}.$$

$\therefore$ $\hat{\theta}$ is the unique maximiser. $\square$

(c) Note that $n\hat{\theta} = \sum_{i=1}^{n}(X_i - X_{(1)})$

$$= \sum_{j=2}^{n}(X_{(j)}-X_{(j-1)})(n-j+1)$$

$$= \sum_{j=2}^{n} Z_j\,.$$

$$\therefore \quad \frac{2n\hat{\theta}}{\theta} = \sum_{j=2}^{n}\frac{2Z_j}{\theta}$$

So it suffices to show that $\frac{2Z_j}{\theta} \overset{iid}{\sim} \chi^2_2 \overset{d}{=} \text{Gamma}(\overset{\text{shape}}{\overset{\downarrow}{\tfrac{2}{2}}}, \overset{\text{rate}}{\overset{\downarrow}{\tfrac{1}{2}}})$

or equivalently, that $Z_j \overset{iid}{\sim} \frac{\theta}{2}\chi^2_2 \overset{d}{=} \text{Gamma}(1, \frac{1}{\theta}) \overset{d}{=} \text{Exp}(\frac{1}{\theta})$.

To this end, compute:

$P(Z_j > z_j \ \forall j = 2,\dots,n)$ ~~$= \sum_{\substack{\text{permutations}\\ \pi \text{ of } \{1,\dots,n\}}} P(Z_j > z_j\ \forall j \mid X_{(1)} = X_{\pi(1)},\dots,X_{(n)} = X_{\pi(n)})\,P(X_{(1)} = X_{\pi(1)},\dots,X_{(n)})$~~

$$= \sum_{\substack{\text{permutations}\\ \pi \text{ of } \{1,\dots,n\}}} P(Z_j > z_j \forall j \mid X_{(1)} = X_{\pi(1)},\dots,X_{(n)} = X_{\pi(n)})\,P(X_{(1)} = X_{\pi(1)},\dots,X_{(n)} = X_{\pi(n)})$$

$$= \frac{1}{n!}\sum_{\pi} P(Z_j > z_j \forall j \mid X_1 < X_2 < \dots < X_n)$$

$$= P(Z_j > z_j \ \forall j \mid X_1 < X_2 < \dots < X_n)$$

~~$= P(Z_j > z_j\ \forall j$~~

$$= P(Z_2 > z_2,\dots, Z_n > z_n \mid X_1 < \dots < X_n)$$

1997 Q7

$$P\left(X_{(j)}-X_{(j-1)}>\frac{z_j}{n-j+1}\ \forall j\geq 2\right)=$$

$$=n!\,P\left(X_j-X_{j-1}>\frac{z_j}{n-j+1}\ \forall j\geq 2\right)$$

$$=n!\int_0^\infty P\left(X_j-X_{j-1}>\frac{z_j}{n-j+1}\ \forall j\geq 2\,\middle|\,X_1=x_1\right)\frac{e^{-\frac{x_1}{\theta}}}{\theta}\,dx_1$$

$$=n!\int_0^\infty\left\{P\left(X_j>\frac{z_j}{n-j+1}+X_{j-1}\ \forall j\geq 2\,\middle|\,X_1=x_1\right)\right\}\frac{e^{-\frac{x_1}{\theta}}}{\theta}\,dx_1$$

$$=n!\int_0^\infty\left\{\int_{x_1+\frac{z_2}{n-1}}^\infty\int_{x_2+\frac{z_3}{n-2}}^\infty\int_{x_3+\frac{z_4}{n-3}}^\infty\cdots\int_{x_{n-1}+z_n}^\infty\frac{1}{\theta^{n-1}}e^{-\frac{x_2}{\theta}-\frac{x_3}{\theta}-\cdots-\frac{x_n}{\theta}}\,dx_n\,dx_{n-1}\cdots dx_2\right\}\frac{e^{-\frac{x_1}{\theta}}}{\theta}\,dx_1$$

$$=n!\int_0^\infty\left\{e^{-\frac{z_2}{\theta}}e^{-\frac{z_3}{\theta}}\cdots e^{-\frac{z_n}{\theta}}\right\}\frac{e^{-\frac{x_1}{\theta}}}{\theta}\,dx_1$$

$$=n!\,e^{-\frac{z_2}{\theta}}\cdots e^{-\frac{z_n}{\theta}},$$

as was required to show. Alternatively, $\underbrace{\Sigma(X_i-X_{(1)})}_{?}+\underbrace{n(X_{(1)})}_{\text{indep }\operatorname{Gamma}(1,\frac{1}{\theta})}=\Sigma X_i\sim\text{indep}+\operatorname{Gamma}(n,\frac{1}{\theta})$

By Basu's, $\Sigma(X_i-X_{(1)})\perp\!\!\!\perp X_{(1)}$ $\therefore$ $\Sigma(X_i-X_{(1)})\sim\operatorname{Gamma}(n-1,\frac{1}{\theta})$ by MGF argument.

(d) By the previous part,

~~$\frac{2n\hat{\theta}}{\theta}\overset{d}{=}\sum_1^{2(n-1)}Y_i$ where $Y_i\overset{iid}{\sim}\chi^2_1$~~

$$\frac{2n\hat{\theta}}{\theta}=\sum_2^n\frac{2z_j}{\theta}$$

and $\frac{2z_j}{\theta}\overset{iid}{\sim}\chi^2_2$

$$\therefore\ \sqrt{n}\left(\Sigma\frac{2z_j}{\theta n}-2\right)\xrightarrow{d}N(0,4)$$

$$\therefore\ \sqrt{n}\left(\frac{2}{\theta}\hat{\theta}-2\right)\xrightarrow{d}N(0,4)$$

$\therefore \quad \sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \theta^2)$

and so asymptotically $\hat{\theta}_n \approx N(\theta, \frac{\theta^2}{n})$ and

$\mathrm{Var}\, \hat{\theta}_n \cancel{=} \sim \frac{\theta^2}{n}$ $\square$

1997 Q1

(a) $L(\theta; X) = \frac{1}{\theta^n} \mathbb{1}\{\theta > X_{(n)}\}$

$$\therefore \frac{L(\theta; X)}{L(\theta; Y)} = \frac{\mathbb{1}\{\theta > X_{(n)}\}}{\mathbb{1}\{\theta > Y_{(n)}\}}$$

which is clearly independent of θ iff $X_{(n)} = Y_{(n)}$.
iff $T(\vec{X}) = T(\vec{Y})$

when $T(X) = X_{(n)}$, ~~to min~~ ~~$= X_{(1)}$~~

$\therefore T(X) = X_{(n)}$ is M.S.

(b) By class results, $X_{(n)}$ is also complete

Also $E X_{(n)} = \frac{n}{n+1}\theta$ $\therefore$ $\frac{n+1}{3n} X_{(n)}$ is UMVUE for $g(\theta)$ □

(c) By NP lemma, an MP test is

$$\phi(X) = 1 \quad \text{if} \quad \cancel{\text{if } \theta_1} \; p_{\theta_1}(X) > k \cdot p_{\theta_0}(X)$$
$$= 0 \quad \text{if} \quad p_{\theta_1}(X) < k \cdot p_{\theta_0}(X)$$

$E_{\theta_0} \phi(X) = \alpha.$

Pick $k = \left(\frac{\theta_0}{\theta_1}\right)^n$ then we have the test

$$\phi(X) = 1 \quad \text{if} \quad X_{(n)} \geq \theta_0$$
$$= \alpha \quad \text{o/w}$$

satisfies the ~~⇒~~ NP lemma $\therefore$ it is ~~mod~~ MP □

(d) Our test ϕ is free of the alternative $\theta = \theta_1 > \theta_0$

$\therefore$ ϕ is UMP for $\theta = \theta_0$ vs $\theta > \theta_0$. $\square$

1997 Q2

(a) $$L(\lambda; x) = \prod_{i=1}^{n} \frac{e^{-\alpha_i \lambda} \lambda^{x_i} \alpha_i^{x_i}}{x_i!}$$

$$= e^{-\sum \alpha_i \lambda} \lambda^{\sum x_i} \prod \frac{\alpha_i^{x_i}}{x_i!}$$

$$\therefore \ell(\lambda; x) = -\lambda \sum \alpha_i + (\sum x_i) \log \lambda + \text{constant} \quad \text{Ⓘ}$$

$$\therefore \ell'(\lambda; x) = -\sum \alpha_i + \frac{\sum x_i}{\lambda}$$

$$\therefore \ell''(\lambda; x) = -\frac{\sum x_i}{\lambda^2} < 0 \quad \text{Ⓘ}$$

$\therefore \ell$ is concave and has a unique maximum at $\ell' = 0$

$$\therefore \text{MLE is } \hat{\lambda}_n = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} \alpha_i} \quad \square$$

(b) By additivity of the Poisson

$$\sum_{1}^{n} X_i \sim \text{Poisson}\left(\lambda \sum_{i=1}^{n} \alpha_i\right)$$

~~$\therefore \phi_{\Sigma X_i}(t) =$~~

Note that $\phi_{X_j}(t) = E\, e^{itX_j} = e^{\lambda(e^{it}-1)}$

$$\therefore \phi_{\sum_1^n X_i}(t) = e^{\lambda(\sum_1^n \alpha_i)(e^{it}-1)}$$

$$\therefore \phi_{\hat{\lambda}_n}(t) = \phi_{\Sigma X_i}\left(\frac{t}{\Sigma \alpha_i}\right) = e^{\lambda(\Sigma \alpha_j)(e^{it/\Sigma \alpha_j}-1)}$$

$$\therefore \phi_{\hat{\lambda}_n - \lambda}(t) = e^{-it\lambda} e^{\lambda(\Sigma \alpha_j)(e^{it/\Sigma \alpha_j}-1)}$$

$\therefore \quad \phi_{\sqrt{\Sigma\alpha_j}(\hat{\lambda}_n-\lambda)}(t) = e^{-i\lambda\sqrt{\Sigma\alpha_j}\,t}\; e^{\lambda(\Sigma\alpha_j)(e^{it/\sqrt{\Sigma\alpha_j}}-1)}$

~~log ϕ~~ Let $T_n = \sqrt{\sum_1^n \alpha_j}\,(\hat{\lambda}_n - \lambda)$, $s_n = \sum_1^n \alpha_j$

$\phi_{T_n}(t) = \exp\{-i\lambda t\sqrt{s_n} + \lambda s_n(e^{it/\sqrt{s_n}} - 1)\}$

$$\phi_{T_n}(t) = \exp\{-i\lambda t\sqrt{s_n} + \lambda s_n(it/\sqrt{s_n} + O(\tfrac{1}{s_n}))\}$$

$$= \exp\{-i\lambda t\sqrt{s_n} + \lambda s_n(it/\sqrt{s_n} + i^2t^2/(2s_n) + O(s_n^{-\frac{3}{2}})\}$$

$$= \exp\{-\tfrac{1}{2}\lambda t^2 + O(s_n^{-\frac{1}{2}})\}$$

$$\longrightarrow \exp\{-\tfrac{1}{2}\lambda t^2\}$$ as we assume $s_n \to \infty$. $\therefore$

$T_n \xrightarrow{d} N(0,\lambda)$ $\square$. Alternatively, use Lyapunov's CLT.

(c) From II, the information in our sample is

$I(\lambda) = \frac{\Sigma\alpha_i}{\lambda}$ if $\sum_1^\infty \alpha_i < \infty$, the information does not $\to \infty$.

Alternatively, fixing $\lambda_2 > \lambda_1$, ~~$\ell(\lambda_2;$~~ from I,

$$\ell_n(\lambda_2; x) - \ell_n(\lambda_1; x) = (\lambda_2 - \lambda_1)\sum_1^n \alpha_j + (\log\tfrac{\lambda_2}{\lambda_1})\sum_1^n X_j$$
$$= (\lambda_2-\lambda_1)\sum_1^n \alpha_j + (\log\tfrac{\lambda_2}{\lambda_1})(\Sigma X_j - \alpha_j\lambda_1) + \lambda(\log\tfrac{\lambda_2}{\lambda_1})\sum_1^n \alpha_j$$

But $E_{\lambda_1}\sum_1^n X_j - \alpha_j\lambda_1 = 0$, $\mathrm{Var}_{\lambda_1}\Sigma(X_j - \alpha_j\lambda_1) = \lambda\sum_1^n \alpha_j \le s_\infty \;\forall n$

$\therefore$ $\ell_n(\lambda_2; x) - \ell_n(\lambda_1; x)$ is tight i.e. converges along a subsequence (Prohorov)

$\therefore$ $P^{(n)}_{\lambda_1} \lhd P^{(n)}_{\lambda_2}$ by Le Cam's first lemma. $\square$

1997 Q3

1. $$L(\theta;y) = \prod_{i=1}^{n} \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(y_i-\mu_1)^2}{2\sigma_1^2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(y_i-\mu_2)^2}{2\sigma_2^2}}$$

$$= \frac{1}{2\sqrt{2\pi}} \cdot \prod_{i=1}^{n} \left(\frac{1}{\sigma_1} e^{-\frac{(y_i-\mu_1)^2}{2\sigma_1^2}} + \frac{1}{\sigma_2} e^{-\frac{(y_i-\mu_2)^2}{2\sigma_2^2}} \right)$$

Therefore, if we choose $\mu_1 = y_j$ for some fixed j,

and ~~σ~~ let $\sigma_1 \downarrow 0$, $L(\theta;Y) \to \infty$, so that

$(\mu_1, \sigma_1) = (y_j, 0)$ maximizes the likelihood.

Similarly, $(\mu_2, \sigma_2) = (y_j, 0)$ maximizes the likelihood.
$\forall$ other values, $L(\theta;Y) < \infty$.
$\therefore$ the answer is: $\{\theta : (\mu_1,\sigma_1) = (y_j, 0)$ or $(\mu_2, \sigma_2) = (y_j, 0)$ for some $j\}$

2. Clearly $X_i \not\to \mu_1$ and $0 \not\to \sigma_1^2$ $\forall$ μ_1, σ_1^2 $\square$

3. Theorem requires that the likelihood eqn have a unique root, which is clearly not the case here.

1997 Q5

1. Note that $R(\theta)$

$$R(\delta,\theta) = E_\theta(\delta(X)-\mu-(\hat{Y}_m-\mu))^2$$

$$= E_\theta(\delta(X)-\mu)^2 - 2E(\delta(X)-\mu)(\hat{Y}_m-\mu) + E(\hat{Y}_m-\mu)^2$$

$$= E_\theta(\delta(X)-\mu)^2 + \frac{\sigma^2}{m} \qquad \text{(independence)}$$

As $\frac{\sigma^2}{m}$ is a constant, this amounts to ~~minimising~~ finding UMVUE for μ.

~~As $\bar{X}$ is an~~ $\delta^*(X) = \bar{X}$, as this is an unbiased function of the C.S. statistic.

It is unique, as there is only 1 unbiased function of the C.S. statistic

2. We note that $\hat{Y}_m \sim N(\mu, \frac{\sigma^2}{m})$, $\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$

$$\therefore \quad \hat{Y}_m - \bar{X}_n \sim N(0, \frac{\sigma^2}{m}+\frac{\sigma^2}{n}) = N(0, \sigma^2(\frac{1}{m}+\frac{1}{n}))$$

$\therefore \quad \dfrac{\hat{Y}_m - \bar{X}_n}{\sqrt{\frac{1}{m}+\frac{1}{n}}} \sim N(0,\sigma^2)$. Letting $S^2 = \frac{1}{(n-1)}\sum_i (X_i-\bar{X})^2$,

$$\therefore \quad \frac{\hat{Y}_m - \bar{X}_n}{\sigma\sqrt{\frac{1}{m}+\frac{1}{n}}} \Big/ \sqrt{S^2/\sigma^2} \sim t_{n-1}$$

$$\therefore \quad \frac{\hat{Y}_m - \bar{X}_n}{S\sqrt{\frac{1}{m}+\frac{1}{n}}} \sim t_{n-1}$$

∴ A good prediction interval that is symmetric is:

$$P\left(t_{n-1,\frac{\alpha}{2}} < \frac{\hat{Y}_m - \bar{X}}{S\sqrt{\frac{1}{m}+\frac{1}{n}}} < t_{n-1,1-\frac{\alpha}{2}} \right) = 1-\alpha$$

$$\therefore P\left(\underbrace{\bar{X} - S\sqrt{\frac{2}{m}+\frac{1}{n}}\, t_{n-1,1-\frac{\alpha}{2}}}_{a(X)} < \hat{Y}_m < \underbrace{\bar{X} + S\sqrt{\frac{1}{m}+\frac{1}{n}}\, t_{n-1,1-\frac{\alpha}{2}}}_{b(Y)} \right)$$

1997 Q6

As ~~[illegible]~~ $\psi(x,\theta)$ is non-increasing in θ, $M(\theta)$ is also non-increasing. As $M(\theta_0)=0$ and $M'(\theta_0)\neq 0$, it follows that θ_0 is the unique root of $M(\theta)$.

Using (i) and (ii), fix $\varepsilon>0$ s.t.

I. $M(\theta)$ is strictly decreasing on $[\theta_0-\varepsilon,\theta_0+\varepsilon]$
II: $\int \psi^2(x-\theta)\,dF(x) < \infty \quad \forall \theta\in[\theta_0-\varepsilon,\theta_0+\varepsilon]$

By II $\frac{M_n(\theta)}{n} \xrightarrow{p} E\,\psi(X_i-\theta) = M(\theta)$, $\forall\theta\in[\theta_0-\varepsilon,\theta_0+\varepsilon]$.

~~($\therefore \frac{M_n(\theta)}{n} - M(\theta) \xrightarrow{p} 0 \quad \forall\theta\in[\theta_0-\varepsilon,\theta_0+\varepsilon]$~~
~~$\therefore \sup_{\theta\in[\theta_0-\varepsilon,\theta_0+\varepsilon]} \left|\frac{M_n(\theta)}{n}-M(\theta)\right| \xrightarrow{p} 0$~~
~~(convergence on compact set ⇒ uniform convergence).)~~

But $M_n(\theta)$ is non-increasing (sum of non-increasing fns)

$\therefore$ w.h.p. $\frac{1}{n} M_n(\theta_0-\varepsilon) > 0 > \frac{1}{n} M_n(\theta_0+\varepsilon)$
$\therefore$ w.h.p. $\hat{\theta}_n\in[\theta_0-\varepsilon,\theta_0+\varepsilon]$

~~Repeating the argument~~
As the argument holds ∀ sufficiently small ε, we have that $\hat{\theta}_n \xrightarrow{p} \theta_0$ □

1997 Q7

Let $[a,b]$ satisfy (a)(b)(c), and suppose

$[x,y]$ is some other interval satisfying (a).

~~We claim $x \le y$~~ We are asked to show $y - x \geq b - a$.

Case 1: $x \le a$.

If $y \geq b$, we are done. □

~~Otherwise~~ If $a < y < b$, ~~then~~ $f(x) \le f(a) = f(b) \le f(y)$

and $\int_x^y f(t)\,dt \le \int_{x+\varepsilon}^{y+\varepsilon} f(t)\,dt$ where $\varepsilon = \min(a-x, b-y)$

If $a - x < b - y$, then $\varepsilon = a - x$ and $y + \varepsilon = y + a - x < b$

$$\therefore \quad 1 - \alpha \le \int_a^{y+a-x} f(t)\,dt < \int_a^b f(t)\,dt = 1 - \alpha,$$

a contradiction. Therefore $a - x \geq b - y$ i.e. $y - x \geq b - a$ □

~~Case 2: $x > a$, $x < b$~~

~~If $y \le b$, we have $1 - \alpha = \int_x^y f(t)\,dt < \int_a^b f(t)\,dt = 1 - \alpha$.~~

~~If $y > b$, we have $\int$~~

Lastly, if $y < a$, then $f(x) \le f(y) \le f(a) = f(b)$ so

$$1-\alpha = \int_x^y f(t)\,dt \le \int_{x+b-y}^{b} f(t)\,dt$$

And if $x+b-y > a$, then ~~RHS > ∫~~ RHS $< \int_a^b f(t)\,dt = 1-\alpha$ $\times$.

$\therefore$ $x+b-y \le a$ i.e. $y-x \ge b-a$ $\square$.

<u>Case 2</u>: $x > a$.

If $y \le b$, then $1-\alpha = \int_x^y f(t)\,dt > \int_a^b f(t)\,dt = 1-\alpha$ $\times$.

Otherwise, $y > b$. In this case, ~~$f(b) \le f($~~

$f(x) \ge f(y)$, and [illegible]

$$1-\alpha = \int_x^y f(t)\,dt \le \int_a^{y-(x-a)} f(t)\,dt = \int_a^{y-x+a} f(t)\,dt$$

If $y-x+a < b$, then RHS $< \int_a^b f(t)\,dt$ $\times$

Otherwise, ~~then~~ $y-x+a \ge b$ so $y-x \ge b-a$ $\square$.

1996 Q1

$$f(x) = \frac{\theta}{2} e^{-\theta|x|} \qquad x \in \mathbb{R}.$$

$$p_\theta(\vec{x}) = \frac{\theta^n}{2^n} e^{-\theta \sum_{i=1}^{n} |x_i|}$$

this is an exponential family natural parameter $-\theta$

C.S. statistic is $T(X) = \sum_{i=1}^{n} |X_i|$

Note that $P(|X_1| > 1) = 2\int_1^\infty \frac{\theta}{2} e^{-\theta x}\, dx = \left[-e^{-\theta x}\right]_1^\infty = e^{-\theta}$

Therefore $\delta(X) = \mathbb{1}_{\{|X_1|>1\}}$ is an unbiased estimator of $\exp(-\theta)$.

By Rao Blackwell, $\delta_0 = E\,\delta \,|\, T$ is UMVUE.

rate

Now, the distn. of $|X_i| \overset{iid}{\sim} \text{Exp}(\theta) = \text{Gamma}(1, \theta)$

$\therefore \quad \sum_1^n |X_i| \sim \text{Gamma}(n, \theta)$, and

ancillary to θ c.s. for θ

Basu's thm.

$|X_1| \perp\!\!\!\perp \sum_2^n |X_i| \sim \text{Gamma}(n-1, \theta)$

and $\frac{|X_1|}{\sum_1^n |X_i|} \sim \text{Beta}(1, n-1) \perp\!\!\!\perp \sum_1^n |X_i|$

Thus, ~~$|X_1|$~~ ~~$E\delta$~~ $|T$

$$E[\delta \,|\, T=t] = E\left[\mathbb{1}_{\{|X_1|>1\}} \,\middle|\, \sum_1^n |X_i| = t\right] = E\left[\mathbb{1}_{\left\{\frac{|X_1|}{\sum |X_i|} > \frac{1}{t}\right\}} \,\middle|\, \sum_1^n |X_i| = t\right]$$

$$= P\left(\text{Beta}(1, n-1) > \frac{1}{t}\right) = \int_{1/t}^{1} \frac{1}{n-1} (1-s)^{n-2}\, ds = \left[(1-s)^{n-1}\right]_{1/t}^{1} = \left(1 - \frac{1}{t}\right)^{n-1}$$

$\therefore$ UMVUE is

$$\delta_0 = \int_{1/t}^{1}$$

~~$\delta_0(\vec{X}) = \begin{cases} 1 & \text{if } \sum_1^n |X_i| < 1 \\ \int_{1/\sum |X_i|}^{1} \beta(1, n-1)\, (1-s)^{n-2}\, ds \end{cases}$~~

$$\delta_0(\vec{X}) = \begin{cases} 0 & \text{if } \sum_1^n |X_i| < 1 \\ \left(1 - \frac{1}{\sum_1^n |X_i|}\right)^{n-1} & \text{if } \sum_1^n |X_i| > 1 \end{cases}$$

Alternatively, can also compute

$P(|X_1| \geq 1 \mid \sum |X_i| = t)$ by noting

$$\bar{P}(|X_1| = x \mid \sum |X_i| = t) = \frac{\theta^n e^{-\theta x} (t-x)^{n-2} e^{-\theta(t-x)} \frac{1}{(n-2)!}}{\frac{\theta^n}{(n-1)!} t^{n-1} e^{-\theta t}}$$

$$= (n-1) \frac{1}{t^{n-1}} (t-x)^{n-2}$$

$$\therefore P(|X_1| \geq 1 \mid \sum |X_i| = t) = \int_1^t (n-1) \frac{1}{t^{n-1}} (t-x)^{n-2}\, dx$$

$$= \left(1 - \frac{1}{t}\right)^{n-1}$$

1996 Q2

(a) ~~$L(\mu_1,\mu_2; X) =$~~

Write (X_i, Y_i, Z_i) for $(X_{1_i}, X_{2_i}, X_{3_i})$.

$$L(\mu_1,\mu_2; X, Y) = (2\pi)^{-n} \exp\left\{-\tfrac{1}{2}\Sigma(X_i-\mu_1)^2 - \tfrac{1}{2}\Sigma(Y_i-\mu_2)^2\right\}$$

$$= (2\pi)^{-n} \exp\left\{-\frac{n}{2}(\mu_1^2+\mu_2^2) + n\bar{X}\mu_1 + n\bar{Y}\mu_2 - \tfrac{1}{2}\Sigma X_i^2 - \tfrac{1}{2}\Sigma Y_i^2\right\}$$

Let $\theta_1 = \mu_1 - \mu_2$ $\quad \theta_2 = \mu_1 + \mu_2$, (bijective map $\mathbb{R}^2 \to \mathbb{R}^2$)

so that $\mu_1 = \frac{\theta_1+\theta_2}{2}$, $\mu_2 = \frac{\theta_2-\theta_1}{2}$, $\mu_1^2+\mu_2^2 = \frac{1}{2}(\theta_1^2+\theta_2^2)$

~~$\therefore L(\mu_1,\mu_2$~~

$$\therefore L(\theta_1,\theta_2; X, Y) = (2\pi)^{-n} \exp\left\{-\frac{n}{4}(\theta_1^2+\theta_2^2) + \frac{n\bar{X}+n\bar{Y}}{2}\theta_2 + \frac{n\bar{X}-n\bar{Y}}{2}\theta_1 - \tfrac{1}{2}\Sigma(X_i^2+Y_i^2)\right\}$$

We recognise a 2-parameter exponential family with C.S. statistic

$T_1 = \frac{n\bar{X}-n\bar{Y}}{2}$ $\quad T_2 = \frac{n\bar{X}+n\bar{Y}}{2}$.

Our test $H_0: \mu_1 = \mu_2$ vs $H_1: (\mu_1,\mu_2) \in \mathbb{R}^2$ is equivalent to

$H_0: \theta_1 = 0, \theta_1 \in \mathbb{R}$ vs $H_1: (\theta_1,\theta_2) \in \mathbb{R}^2$.

By class results, $\exists$ a UMPU test of the form

$$\begin{aligned}\phi(X,Y) &= 1 \quad \text{if} \quad \cancel{T_1 \notin [c_1, c_2]} \quad T_1(X,Y) \notin [c_1(T_2), c_2(T_2)] \\ &= 0 \quad \text{if} \quad T_1(X,Y)\ \cancel{\notin} \in (c_1(T_2), c_2(T_2)) \\ &= \gamma_i(T_2) \quad \text{if} \quad T_1(X,Y) = c_i(T_2)\end{aligned}$$

where $E_{\theta_1=0;\theta_2}\left[\phi(X,Y)\,\middle|\,T_2\right]=\alpha$ a.s.

and $E_{\theta_1=0,\theta_2}\left[\phi(X,Y)T_1(X,Y)\,\middle|\,T_2\right]=\alpha\,E_{\theta_1=0,\theta_2}\left[T_1(X)\,\middle|\,T_2\right]$ a.s.

The first level constraint requires

$$P_{\theta_1=0,\theta_2}\left(\bar{X}-\bar{Y}\notin[\tilde{c}_1(T_2),\tilde{c}_2(T_2)]\,\middle|\,T_2\right)=\alpha$$

and the second, that

$$E_{\theta_1=0,\theta_2}\left[\cancel{\phi}\;\mathbb{1}\{\bar{X}-\bar{Y}\notin[\tilde{c}_1(T_2),\tilde{c}_2(T_2)]\}\left(\frac{n\bar{X}-n\bar{Y}}{2}\right)\,\middle|\,T_2\right]=\alpha\,E_{\theta_1=0,\theta_2}\left[\frac{n\bar{X}-n\bar{Y}}{2}\,\middle|\,T_2\right]$$

now note that $\mathrm{Cov}(\bar{X}-\bar{Y},\bar{X}+\bar{Y})=\mathrm{Var}\,\bar{X}-\mathrm{Var}(\bar{Y})=0$

alternatively, use Basu's here on sub-parameter space $\theta_1=0,\ \theta_2\in\mathbb{R}$

and as $(\bar{X},\bar{Y})$ are MVN; $T_1 \perp\!\!\!\perp T_2$. (Also T_2 is sufficient for θ_2

so the expectations above are independent of θ_2). Therefore,

we can remove the conditioning to find:

$$\cancel{E}\;P_{\theta_1=0}\left(\bar{X}-\bar{Y}\notin[\tilde{c}_1(T_2),\tilde{c}_2(T_2)]\,\middle|\,T_2\right)=\alpha \quad \text{a.s.}$$

$$E_{\theta_1=0}\left[\mathbb{1}\{\bar{X}-\bar{Y}\notin[\tilde{c}_1(T_2),\tilde{c}_2(T_2)]\}\left(\frac{n\bar{X}-n\bar{Y}}{2}\right)\,\middle|\,T_2\right]=\alpha\,E_{\theta_1=0}[T_1]=0 \quad \text{a.s.}$$

~~E~~ $\therefore$ for almost all t,

$$E_{\theta_1=0}\left[\mathbb{1}\{\bar{X}-\bar{Y}\notin[\tilde{c}_1(t),\tilde{c}_2(t)]\}\left(\frac{n\bar{X}-n\bar{Y}}{2}\right)\right]=0$$

1996 Q2

equivalently, ~~$E[\mathbb{1}$~~ { letting $z = \bar{X} - \bar{Y} \sim N(0, \frac{2}{n}\cancel{\sigma^2})$ under $\theta_1 = 0$,

$$E\left[z \, \mathbb{1}\{z \notin [\tilde{c}_1(t), \tilde{c}_2(t)]\} \right] = 0$$

By symmetry, it follows that

$$\tilde{c}_1(t) = -\tilde{c}_2(t)$$

Then, from the first level constraint, for almost all t,

$$P_{\theta_1 = 0}\left(z \notin [-\tilde{c}_2(t), \tilde{c}_2(t)] \right) = \alpha \quad \text{so} \quad \frac{\tilde{c}_2(t)}{\sqrt{\frac{2}{n}\cancel{\sigma^2}}} = z_{1-\frac{\alpha}{2}}$$

$\therefore$ our UMPU test is in fact:

$$\begin{aligned} \phi(X,Y) &= 1 \quad \text{if} \quad \bar{X} - \bar{Y} \notin \left[-z_{1-\frac{\alpha}{2}} \sqrt{2/n} \, , \, z_{1-\frac{\alpha}{2}} \sqrt{2/n} \right] \\ &= 0 \quad \text{if} \quad \bar{X} - \bar{Y} \in (\text{ditto}) \\ &= 1 \ \text{(or anything)} \quad \text{if} \ \bar{X} - \bar{Y} = \pm z_{1-\frac{\alpha}{2}} \sqrt{2/n} . \end{aligned}$$

(b) No, there is no UMP for this problem as we have a 2-sided alternative in a normal family.

Suppose $\psi(X,Y)$ was UMP. Then $\psi(X,Y)$ is also UMPU, so it has the same power function as ϕ.

But the test
$$\begin{aligned} \tilde{\psi} &= 1 \quad \text{if} \quad \bar{X} - \bar{Y} > z_{1-\alpha} \sqrt{2n} \\ &= 0 \quad \text{o/w} \end{aligned}$$

is level α and

has greater power when $\mu_1 > \mu_2$ ... $\cancel{X}$

$\therefore$ No UMP test exists here.

(c) The MLE under $\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0$ is

$$\hat{\mu}_1 = \bar{X}_+ = \max(0, \bar{X}), \quad \hat{\mu}_2 = \bar{Y}_+, \quad \hat{\mu}_3 = \bar{Z}_+$$

$$\Lambda^{-1} = \frac{\sup_{H_0 \cup H_1} L(\mu_1, \mu_2, \mu_3; x, y, z)}{\sup_{H_0} L(\mu_1, \mu_2, \mu_3; x, y, z)} = \exp\left\{-\frac{1}{2}\left[\Sigma(X_i - \bar{X}_+)^2 + \Sigma(Y_i - \bar{Y}_+)^2 + \Sigma(Z_i - \bar{Z}_+)^2 - \Sigma X_i^2 + Y_i^2 + Z_i^2\right]\right\}$$

$$= \exp\left\{+\frac{1}{2}\left[2n\bar{X}\bar{X}_+ + 2n\bar{Y}\bar{Y}_+ + 2n\bar{Z}\bar{Z}_+ - n\bar{X}_+^2 - n\bar{Y}_+^2 - n\bar{Z}_+^2\right]\right\}$$

$$= \exp\left\{+\frac{n}{2}\left[\bar{X}_+^2 + \bar{Y}_+^2 + \bar{Z}_+^2\right]\right\}$$

$$\therefore -2\log\Lambda = n\left(\bar{X}_+^2 + \bar{Y}_+^2 + \bar{Z}_+^2\right)$$

$$\cancel{= \left(\frac{\bar{X}_+}{\sqrt{n}}\right)^2 + \left(\frac{\bar{Y}_+}{\sqrt{n}}\right)}$$

$$= (\sqrt{n}\bar{X}_+)^2 + (\sqrt{n}\bar{Y}_+)^2 + (\sqrt{n}\bar{Z}_+)^2$$

Now note, under H_0, $(\sqrt{n}\bar{X}, \sqrt{n}\bar{Y}, \sqrt{n}\bar{Z}) \longrightarrow N(0, I_3)$

$$\therefore (\sqrt{n}\bar{X}_+, \sqrt{n}\bar{Y}_+, \sqrt{n}\bar{Z}_+) \rightsquigarrow \overbrace{(N(0,1)_+, N(0,1)_+, N(0,1)_+)}^{\text{independent}} = (Z_1, Z_2, Z_3) \text{ where } Z_1, Z_2, Z_3 \overset{iid}{\sim} N(0,$$

$$\therefore -\log\Lambda \xrightarrow{d} Z_1^2 + Z_2^2 + Z_3^2 \quad \text{where } Z_1^2, Z_2^2, Z_3^2 \overset{iid}{\sim} \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ \chi_1^2 & \text{w.p. } \frac{1}{2} \end{cases}$$

$$= \begin{cases} 0 & \text{w.p. } 1/8 \\ \chi_1^2 & \text{w.p. } 3/8 \end{cases}$$

1996 Q4

(a) $F_0(X_1), \dots, F_0(X_n) \overset{iid}{\sim} U(0,1)$ as F_0 is cts.

As F_0 is monotone it preserves ordering, so

$$F_0(X_{(1)}), \dots, F_0(X_{(n)}) \overset{d}{=} U_{(1)}, \dots, U_{(n)}$$

where $U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$.

$$\therefore \ E\, F_0(X_{(s)}) - F_0(X_{(r)}) = E\, U_{(s)} - E\, U_{(r)}$$

$$= \frac{s}{n+1} - \frac{r}{n+1} = \frac{s-r}{n+1}$$

By class results since $U_{(k)} \sim \text{Beta}(k, n-k+1)$.

(b) We seek to evaluate $E\, F_0(X_{(s)}) F_0(X_{(r)}) = E\, U_{(s)} U_{(r)}$

By class results, the joint pdf of $U_{(s)}$ and $U_{(r)}$ is:

$$f_{U_{(r)}, U_{(s)}}(u,v) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} f(u) f(v) F(u)^{r-1} (F(v) - F(u))^{s-r-1} (1-F(v))^{n-s}$$

$$= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} u^{r-1} (v-u)^{s-r-1} (1-v)^{n-s}$$

$$\therefore \ E\, U_{(s)} U_{(r)} = \int_0^1 \int_0^v \frac{n!}{(r-1)!(s-r-1)!(n-s)!} u^r (v-u)^{s-r-1} v(1-v)^{n-s} \, du\, dv$$

$$= \int_0^1 \frac{n!\, v(1-v)^{n-s}}{(r-1)!(s-r-1)!(n-s)!} \int_0^1 (vt)^r (v-vt)^{s-r-1} v \, dt\, dv \qquad (u = tv)$$

$$= \int_0^1 \frac{n!\,\cancel{v^{r+2}}\,(1-v)^{n-s}\,v^{s+1}}{(r-1)!\,(s-r-1)!\,(n-s)!} \int_0^1 \overbrace{t^r (1-t)^{s-r-1}}^{\propto \text{ Beta }(r+1,\, s-r) \text{ density}} dt\, dv$$

$$= \int_0^1 \frac{n!\, r!\, (s-r-1)!}{(r-1)!\,(s-r-1)!\,(n-s)!\,(s\cancel{+1})!} (1-v)^{n-s}\, v^{s+1}\, dv$$

$$= \frac{n!\, r}{(n-s)!\,\cancel{}\, s!}\ \text{Beta}\,(n-s+1,\, s+2) \int_0^1 \frac{1}{B(n-s+1,\, s+2)} (1-v)^{n-s}\, v^{s+1}\, dv$$

$$= \frac{n!\, r}{(n-s)!\, s!} \cdot \frac{(n-s)!\,(s+1)!}{(n+2)!}$$

$$= \frac{r(s+1)}{(n+1)(n+2)}$$

Hence $\text{Cov}\left(F_0(X_{(s)}), F_0(X_{(r)})\right) = E\, U_{(s)} U_{(r)} - E\, U_{(s)}\, E\, U_{(r)}$

$$= \frac{r(s+1)}{(n+1)(n+2)} - \frac{r}{n+1} \cdot \frac{s}{n+1}$$

$$= \frac{(rs+r)(n+1) - rs(n+2)}{(n+1)^2 (n+2)}$$

$$= \frac{rsn + rn + rs + r - rsn - 2rs}{(n+1)^2 (n+2)}$$

$$= \frac{r(n-s+1)}{(n+1)^2 (n+2)} \geq 0$$

← answers 2nd part of (c); any 2 order stats are ~~f~~ +vely correlated, which makes sense.

(c) $\text{Cov}\left(U_{(s)} - U_{(r)},\, U_{(s+1)} - U_{(r+1)}\right) = \text{Cov}\left(U_{(s)}, U_{(s+1)}\right) - \text{Cov}\left(U_{(s)}, U_{(r+1)}\right) - \text{Cov}\left(U_{(r)}, U_{(s+1)}\right) + \text{Cov}\left(U_{(r)}, U_{(r+1)}\right)$

$$= \frac{1}{(n+1)^2 (n+2)} \left[s(n-(s+1)+1) + r(n-(r+1)+1) - (r+1)(n-s+1) - r(n-(s+1)+1) \right]$$

$$= \frac{1}{(n+1)^2 (n+2)} \left[sn - s^2 + rn - r^2 - rn + rs - r \cdots - n + s - 1 - rn + rs \right]$$

$$= \frac{1}{(n+1)^2 (n+2)} \left[sn\ \cancel{} - (s-r)^2 + \underbrace{(s-r-1)}_{} - n \right] > 0$$

$(s-1)n > (s-r)^2$

if $U_{(r)}$ and $U_{(s)}$ are far, that forces ~~them~~ $U_{(s+1)}$ to be far, and $U_{(r+1)}$ will be on average

1996 Q.6

(a) As per HW 1, $T = (X_{(1)}, \dots, X_{(n)})$ is M.S.

(b) $\ell(\theta; X) = -\log(1 + (x-\theta)^2)$

$$\therefore \frac{\partial \ell}{\partial \theta} = \frac{2(x-\theta)}{1+(x-\theta)^2}$$

$$\therefore I(\theta) = E\, \frac{4(\theta - X)^2}{(1+(X-\theta)^2)^2}$$

$$= \frac{4}{\pi}\int_{-\infty}^{\infty} \frac{(x-\theta)^2}{(1+(x-\theta)^2)^3}\, dx$$

$$= \frac{4}{\pi}\int_{-\infty}^{\infty} \frac{u^2}{(1+u^2)^3}\, du \qquad (u = x-\theta)$$

$$= \frac{4}{\pi}\int_{-\infty}^{\infty} u \, \frac{u}{(1+u^2)^3}\, du$$

$$= \frac{4}{\pi}\left[\left(-u\,\frac{(1+u^2)^{-2}}{4}\right)\Bigg|_{-\infty}^{\infty} + \frac{1}{4}\int_{-\infty}^{\infty} \frac{1}{(1+u^2)^2}\, du\right]$$

$$= \cancel{\frac{4}{\pi}}\, \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{1}{(1+u^2)^2} + \frac{1}{(1+u^2)^2}\right) du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{1+u^2}{(1+u^2)^2} + \frac{1-u^2}{(1+u^2)^2}\right) du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+u^2}\, du + \frac{1}{2\pi}\left[\frac{u}{1+u^2}\right]_{-\infty}^{\infty}$$

$$= \frac{1}{2}$$

Alternatively:

$$\int_{-\infty}^{\infty} \frac{u^2}{(1+u^2)^3}\, du = 2\int_{0}^{1} t^3 \left(\frac{1}{t}-1\right) \cdot \frac{1}{2}\left(\frac{1}{t}-1\right)^{-\frac{1}{2}} t^{-2}\, dt \qquad \left(t = (1+u^2)^{-1} \therefore u = \sqrt{1/t - 1} \qquad \frac{du}{dt} = \frac{1}{2}\left(\frac{1}{t}-1\right)^{-\frac{1}{2}}\left(-\frac{1}{t^2}\right) \right)$$

$$= \int_0^1 t\left(\frac{1}{t}-1\right)^{1/2} dt$$

$$= \int_0^1 t^{1/2} \, ~~(\ldots)~~ \, (1-t)^{1/2} \, dt$$

$$= \frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(3)}$$

$$= \frac{\left(\frac{1}{2}\Gamma(\frac{1}{2})\right)^2}{2}$$

$$= \frac{\pi}{8}$$

Same as fisher info in $T(X) = (X_{(1)}, \ldots, X_{(n)})$ as $f_\theta(x_{(1)}, \ldots, x_{(n)}) = n! \prod f_\theta(x_{(i)})$ $\therefore$ same log-likelihood is equal up to a constant $\log n!$

(c) See 1992 Q7 (ii)

1996 Q8

Compute the likelihood :

$$L(\theta; X) = \prod_{i=1}^{n} \left(\tfrac{1}{2} + \theta a_i\right)^{x_i} \left(\tfrac{1}{2} - \theta a_i\right)^{1-x_i}$$

$$E_\theta \bar{X} = \frac{1}{n} \Sigma E_\theta X_i = \frac{1}{n} \Sigma \tfrac{1}{2} + \theta a_i = \frac{1}{2} + \theta \frac{\Sigma a_i}{n}.$$

Let $T_n = \dfrac{\bar{X} - \frac{1}{2}}{\sum_1^n a_i / n} = \dfrac{\sum_1^n X_i - \frac{n}{2}}{\sum_1^n a_i}$

then $E_\theta T_n = \theta$ by the above.

Secondly, $\mathrm{Var}\, \bar{X} = \dfrac{1}{n^2} \mathrm{Var}\, \Sigma X_i = \dfrac{1}{n^2} \Sigma \left(\tfrac{1}{2} + \theta a_i\right)\left(\tfrac{1}{2} - \theta a_i\right)$

$$= \frac{1}{n^2} \Sigma \left(\tfrac{1}{4} - \theta^2 a_i^2\right) = \frac{1}{4n} - \theta^2 \frac{\Sigma a_i^2}{n^2}$$

$$= \frac{n/4 - \theta^2 \Sigma a_i^2}{n^2}$$

won't let Var → 0.

$$\mathrm{Var}\, T_n = \frac{1}{\left(\sum_1^n a_i\right)^2} \mathrm{Var}\left(\Sigma X_i\right) = \frac{\Sigma \left(\frac{1}{4} - \theta^2 a_i^2\right)}{\left(\Sigma a_i\right)^2} = \frac{\frac{1}{4} n - \theta^2 \Sigma a_i^2}{\left(\Sigma a_i\right)^2}$$

Alternatively, consider : ~~$T_n = \Sigma a_i X_i$~~

$$E\, \Sigma a_i X_i = \Sigma a_i \left(\tfrac{1}{2} + \theta a_i\right) = \frac{\Sigma a_i}{2} + \theta \Sigma a_i^2$$

~~$E \dfrac{\Sigma a_i X_i}{\Sigma a_i} = \dfrac{\Sigma a_i E X_i}{\Sigma a_i} = \dfrac{\frac{1}{2} \Sigma a_i + \theta \Sigma a_i^2}{\Sigma a_i} = \dfrac{1}{2} + \theta \dfrac{\Sigma a_i^2}{\Sigma a_i}$~~

$\therefore\ E\left[\dfrac{(\Sigma a_i X_i) - \frac{1}{2} \Sigma a_i}{\Sigma a_i^2}\right] = \theta \qquad \mathrm{Var}\,(\,''\,) = \dfrac{1}{(\Sigma a_i^2)^2} \mathrm{Var}\left(\Sigma a_i X_i\right) =$

$$\frac{1}{(\sum a_i^2)^2}\sum a_i^2 \operatorname{Var} X_i = \frac{1}{(\sum a_i^2)^2}\sum a_i^2\left(\tfrac{1}{4}-\theta a_i^2\right) = \frac{\frac{1}{4}\sum a_i^2 - \theta\sum a_i^4}{(\sum a_i^2)^2}$$

$$= \frac{1}{4}\cdot\frac{1}{(\sum a_i^2)} + \theta\,\frac{\sum a_i^4}{(\sum a_i^2)^2}$$

$$\leq \frac{1}{4}\cdot\frac{1}{\sum a_i^2} + \theta\,\frac{M+\sum a_i^2}{(\sum a_i^2)^2}$$

$\left(\sum_1^n a_i^4 \leq \sum_1^n a_i^2 + M\right.$ as $a_i^4 \leq a_i^2$ $\forall i$ big enough s.t. $\left. a_i < 1 \right)$

$\longrightarrow 0$ as $n \to \infty$. in the case $\sum_1^n a_i^2 \longrightarrow \infty$.

Hence $T_n = \dfrac{\sum a_i X_i - \frac{1}{2}\sum a_i}{\sum a_i^2}$ is an unbiased estimator of θ with $\operatorname{Var}(T_n) \to 0$ ∴ T_n is consistent by Chebyshev's.

Thus, if $\sum a_i^2 \to \infty$, $\exists$ a consistent estimator

Conversely, suppose $\sum_1^\infty a_i^2 = K < \infty$. Consider

$$l(\theta, x) = \sum_{i=1}^n x_i \log\frac{\frac{1}{2}+\theta a_i}{\frac{1}{2}-\theta a_i} + \log\left(\tfrac{1}{2}-\theta a_i\right)$$

$$\therefore\; l(\theta;X) - l(0;X) = \sum X_i \log\frac{1+2\theta a_i}{1-2\theta a_i} + \log(1-2\theta a_i)$$

$$= \sum_{i=1}^n\left\{X_i \log(1+2\theta a_i) - X_i\log(1-2\theta a_i)\right\} + \sum\log(1-2\theta a_i)$$

$$= \sum_{i=1}^n X_i\left(4\theta a_i + O(a_i^3)\right) + \sum\left\{(-2\theta a_i) + O(a_i^2)\right\}$$

$$= 2\theta\sum_{i=1}^n a_i(2X_i - 1) + \sum_{i=1}^n O(a_i^2)$$

1996 Q8

And note that the term $\sum_{i=1}^{n} O(a_i^2) \longrightarrow \tilde{K} < \infty$

whereas $\Longrightarrow \sum a_i$

$E_{\theta=0} \sum a_i(2X_i-1) = \sum a_i E(2X_i-1) = 0$

$\mathrm{Var}_{\theta=0} \sum a_i(2X-1) = \sum a_i^2 \mathrm{Var}\, X_i = \frac{1}{4}\sum a_i^2 \le k$.

Hence $\sum_{i=1}^{n} a_i(2X_i-1)$ is tight .

By Prohorov's, $\sum_{i=1}^{n} a_i(2X_i-1) \xrightarrow{d} Z$ along a subsequence,

and so $\dfrac{L(\theta;X)}{L(0;X)} \xrightarrow{d} e^{2\theta Z + \tilde{K}} > 0$ a.s.

Hence $P_0^{(n)} \Delta P_\theta^{(n)}$ by le Cam's 1st lemma.

$\therefore$ $\exists$ no consistent estimator

Suppose $P_\theta^{(n)}(|T_n-\theta|>\varepsilon) \to 0$

then $P_0^{(n)}(|T_n-\theta|>\varepsilon) \to 0$ X .

1995 Q2

(i) The likelihood is

$$L(\theta,\vec{\mu};\vec{X},\vec{Y}) = \prod_{i=1}^{n} (2\pi\theta)^{-\frac{2}{2}} \exp\left\{-\frac{1}{2\theta}(X_i-\mu_i)^2 - \frac{1}{2\theta}(Y_i-\mu_i)^2\right\}$$

$$= (2\pi\theta)^{-n} \exp\left\{-\frac{1}{2\theta}\sum_i (X_i-\mu_i)^2 + (Y_i-\mu_i)^2\right\}$$

$$= (2\pi\theta)^{-n} \exp\left\{-\frac{1}{2\theta}\sum_i \left[(X_i-\frac{X_i+Y_i}{2})^2 + (Y_i-\frac{X_i+Y_i}{2})^2 + 2\left(\frac{X_i+Y_i}{2}-\mu_i\right)^2\right]\right\}$$

$$= (2\pi\theta)^{-n} \exp\left\{-\frac{1}{2\theta}\sum_i \left[2\left(\frac{X_i-Y_i}{2}\right)^2 + 2\left(\frac{X_i+Y_i}{2}-\mu_i\right)^2\right]\right\}$$

$$\therefore \quad \ell(\theta,\vec{\mu};\vec{X},\vec{Y}) = -n\log(2\pi\theta) - \frac{1}{2\theta}\sum_{i=1}^{n}\left[2\left(\frac{X_i-Y_i}{2}\right)^2 + 2\left(\frac{X_i+Y_i}{2}-\mu_i\right)^2\right]$$

~~This is mse~~

To maximise the likelihood, we first maximise in μ_i

as we have quadratic, clearly ~~$\hat{\mu}$~~ $\hat{\mu}_i = \frac{X_i+Y_i}{2}$ irrespective of θ.

$$\therefore \quad \ell(\theta;\hat{\mu};X,Y) = -n\log(2\pi\theta) - \frac{1}{\theta}\sum\left(\frac{X_i-Y_i}{2}\right)^2 \qquad ⒤$$

$$\therefore \quad \cancel{\frac{\partial \ell}{\partial\theta}}\ \frac{\partial}{\partial\theta}\ell(\theta;\hat{\mu};X,Y) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum\left(\frac{X_i-Y_i}{2}\right)^2 \qquad ⅱ$$

$$\therefore \quad \frac{\partial^2}{\partial\theta^2}\ell(\theta,\hat{\mu};X,Y) = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum\left(\frac{X_i-Y_i}{2}\right)^2 \qquad ⅲ$$

Now note:

- From I, $\ell(\theta) \longrightarrow -\infty$ as $\theta \to 0$ or ∞ (as $\sum\left(\frac{X_i-Y_i}{2}\right)^2 > 0$ as
- From II, ~~$\frac{d\ell}{d\theta}$~~ $\ell(\theta)$ has a unique stationary point at

$$\hat{\theta} = \frac{1}{n}\sum\left(\frac{X_i-Y_i}{2}\right)^2$$

Hence $\hat{\theta}$ must be the MLE (as $\ell(\theta)$ is smooth).

(ii) $X_i - Y_i \sim N(0, 2\theta)$ ∴ ~~[illegible]~~

∴ $\frac{X_i - Y_i}{2} \sim N\left(0, \frac{\theta}{2}\right) = \sqrt{\frac{\theta}{2}}\, N(0,1)$

∴ $\left(\frac{X_i - Y_i}{2}\right)^2 \sim \frac{\theta}{2} \chi^2_1$

By WLLN, $\hat{\theta}_n = \frac{1}{n} \sum \left(\frac{X_i - Y_i}{2}\right)^2 \xrightarrow{p} \frac{\theta}{2}$.

(iii) Clearly, $2\hat{\theta}_n \xrightarrow{p} \theta$ (CMT)

(iv) $\sqrt{n}(2\hat{\theta}_n - \theta) = \sqrt{n}\left(\frac{\sum (X_i - Y_i)^2}{2n} - \theta\right) \xrightarrow{d} N(0, 2\theta^2)$

by the CLT, as $\text{Var} \frac{(X_i - Y_i)^2}{2} = \frac{1}{4} \text{Var}(2\theta \chi^2_1) = \theta^2 \cdot \text{Var}(\chi^2_1) = 2\theta^2$

But by the CMT, $2(2\hat{\theta}_n)^2 = 8\hat{\theta}_n^2 \xrightarrow{p} 2\theta^2$

∴ By Slutsky's, $\frac{\sqrt{n}(2\hat{\theta}_n - \theta)}{2\sqrt{2}\,\hat{\theta}_n} \xrightarrow{d} N(0,1)$

∴ $1-\alpha$ asymptotic confidence interval is ~~$\theta \in$~~ ($-z_{1-\frac{\alpha}{2}} < \frac{\sqrt{n}(2\hat{\theta}_n - \theta)}{2\sqrt{2}\,\hat{\theta}_n} < z_{1-\frac{\alpha}{2}}$

i.e. $\theta \in \left(2\hat{\theta}_n \pm \frac{1}{\sqrt{n}} 2\sqrt{2}\, z_{1-\frac{\alpha}{2}} \hat{\theta}_n\right)$

~~(ii)~~ Alternatively, by independence, we know that

$\sum_1^n (X_i - Y_i)^2 \sim 2\theta \chi^2_n$,

1995 Q2

Therefore we can construct an exact $1-\alpha$ C.I. as follows

~~$\sum(X_i-Y_i)^2 - 2n\theta$~~ $\frac{\sum(X_i-Y_i)^2}{2\theta} \sim \chi^2_n$

$$\therefore P\left(\chi^2_{n,\frac{\alpha}{2}} < \frac{\sum(X_i-Y_i)^2}{2\theta} < \chi^2_{n,1-\frac{\alpha}{2}}\right)$$

$$= P\left(\frac{\sum(X_i-Y_i)^2}{2\chi^2_{n,1-\frac{\alpha}{2}}} < \theta < \frac{\sum(X_i-Y_i)^2}{2\chi^2_{n,\alpha/2}}\right)$$

$$= 1-\alpha$$

$\therefore$ exact $1-\alpha$ C.I. is ~~as~~ $\left(\frac{\sum(X_i-Y_i)^2}{2\chi^2_{n,1-\frac{\alpha}{2}}}, \frac{\sum(X_i-Y_i)^2}{2\chi^2_{n,\frac{\alpha}{2}}}\right)$

(v) From III, ~~$E\left(-\frac{\partial^2}{\partial\theta^2}\ell(\theta,\mu_i;x,y)\right) = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum \underbrace{E\left(\frac{X_i-Y_i}{2}\right)^2}_{\frac{\theta}{2}} = \frac{n}{\theta^2} - \frac{n}{\theta^2} = 0$~~

~~and at $\hat{\theta}_{MLE} = \frac{1}{n}\sum\left(\frac{X_i-Y_i}{2}\right)^2$, this is equal to~~

~~$\frac{n^3}{\left(\sum\left(\frac{X_i-Y_i}{2}\right)^2\right)^2}\left(1-\frac{1}{2}\right) = \frac{n^3}{2\left(\sum\left(\frac{X_i-Y_i}{2}\right)^2\right)^2}$~~

$$\frac{\partial^2 \ell}{\partial\theta^2} = +\frac{n}{\theta^2} - \frac{2}{\theta^3}\sum\left[\left(\frac{X_i-Y_i}{2}\right)^2 + \left(\overbrace{\frac{X_i-Y_i}{2}-\mu_i}^{N(0,\frac{\theta}{2})}\right)^2\right]$$

$$\therefore E -\frac{\partial^2 \ell}{\partial\theta^2} = -\frac{n}{\theta^2} + \frac{2n}{\theta^3}\left[\frac{\theta}{2} + \frac{\theta}{2}\right] = -\frac{n}{\theta^2} + \frac{2n}{\theta^2} = \frac{n}{\theta^2}$$

$\therefore$ ~~Fisher~~ But $\mathrm{Var}(\hat{\theta}_n) =$ ~~$\frac{1}{n^2}$~~ $\mathrm{Var}\left(\frac{1}{n}\sum\left(\frac{X_i-Y_i}{2}\right)^2\right)$

$$= \frac{1}{n^2} n \mathrm{Var}\left(\frac{X_i-Y_i}{2}\right)^2$$

$$= \frac{\theta^2}{2n} < \frac{1}{E-\frac{\partial^2\ell}{\partial\theta^2}} = \frac{\theta^2}{n}$$

$\therefore$ expect inconsistent MLE

1995 Q3

(i) $p_\theta(X) = \prod_{1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\Sigma X_i}(1-p)^{1-\Sigma X_i}$

To find an unbiased estimator we can reduce by sufficiency

$T = \Sigma X_i$ is sufficient, $T \sim$ Binomial (n,p)

~~pr~~ $P_p(T=t) = \binom{n}{t} p^t (1-p)^{n-t}$.

Suppose an estimator $\tilde{\delta}(X)$ is unbiased for θ.

then $\delta(T) = E\,\tilde{\delta}(X)|T$ is also unbiased (tower law)

$\therefore \ E_\theta \delta(T) = \theta \quad \forall \theta$

$\therefore \ \sum_{i=0}^{n} \delta(i)\binom{n}{i} p^i (1-p)^{n-i} = \frac{1}{p} \quad \forall p$

But the LHS is a polynomial of order n in p,

whereas the RHS is the reciprocal of p $\times$.

(ii) $T = \min\{n \geq 1 : X_n = 1\}$ then $T \sim$ Geom (p)

$P(T=t) = (1-p)^{t-1} p$. Then $ET = \frac{1}{p}$ □

(iii) Consider $\hat{\theta}_n = \frac{n}{T_n} = \frac{1}{\bar{X}_n}$.

(a) By WLLN, $\frac{T_n}{n} \xrightarrow{P} p$ so by CMT, $\hat{\theta}_n \xrightarrow{P} \frac{1}{p} = \theta$ so $\hat{\theta}_n$ is consistent

Also, by (ii), $\sqrt{n}\left(\frac{T_n}{n} - p\right) \xrightarrow{d} N(0, p(1-p))$

let $g(p) = \frac{1}{p}$ so $g'(p) = -\frac{1}{p^2} \neq 0$

By Δ theorem, $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \frac{1}{p^4} p(1-p)) = N(0, \frac{1-p}{p^3})$

as required.

(iv) The information in one observation is

$f(x) = p^{x_1}(1-p)^{1-x_1}$ $\therefore$

$\therefore$ $l(p; x) = x \log p + (1-x) \log(1-p)$

$$\therefore \frac{\partial l}{\partial p} = \frac{x}{p} + \frac{x-1}{1-p}$$

$$\therefore \frac{\partial^2 l}{\partial p^2} = -\frac{x}{p^2} + \frac{x-1}{(1-p)^2}$$

$$\therefore E\left[-\frac{\partial^2 l}{\partial p^2}\right] = \frac{1}{p} - \frac{p-1}{(1-p)^2} = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$$

$$\therefore I(p) = \frac{1}{p(1-p)}$$

$$\therefore I(\theta) = I(g(p)) = \frac{I(p)}{(g'(p))^2} = \frac{\left(\frac{1}{p(1-p)}\right)}{\left(\frac{1}{p^4}\right)} = \frac{p^3}{1-p}$$

$\therefore \frac{1}{I(\theta)} = \frac{1-p}{p^3}$ and therefore

$\hat{\theta}_n$ is asymptotically efficient $\square$

1995 Q4

(i) We set

$$\alpha = E_{\theta=0}\,\phi(X) = P_{\theta=0}(X_{(1)} \geq c) = (1-c)^n$$

so $\Leftarrow$ $c = 1-\alpha^{1/n}$.

(ii) $\beta_\phi(\theta) = E_\theta\,\phi(X) = P_\theta(X_{(n)} \geq 1 \text{ or } X_{(1)} \geq c)$

Note that, $\forall\, \theta \geq c$, $\beta_\phi(\theta) = 1$. Otherwise, $\theta \in [0, c)$ so

$$P_\theta(X_{(n)} \geq 1 \text{ or } X_{(1)} \geq c) = P_\theta(X_{(n)} \geq 1) + P_\theta(X_{(n)} \leq 1 \text{ and } X_{(1)} \geq c)$$

$$= 1-(1-\theta)^n + P_\theta(X_i \in [c,1]\ \forall i)$$

$$= 1-(1-\theta)^n + (1-c)^n$$

Hence $\beta_\phi(\theta) = \begin{cases} 1-(1-\theta)^n + (1-c)^n & \text{for } \theta \in [0,c) \\ 1 & \text{for } \theta \geq c. \end{cases}$

(iii) Fix $\theta_1 > 0$. We show that ϕ is MP for $\theta = 0$ vs $\theta = \theta_1$, at level α. From (i) we already know ϕ is level α.

- Case 1: $\theta_1 < c$

~~Consequently~~ with $k = 1$, $p_{\theta_1}(x) > k\,p_{\theta_0}(x)$

$$\Rightarrow 1\{X_{(1)} \geq \theta\}\, 1\{X_{(n)} \leq \theta+1\} > 1\{X_{(1)} \geq 0\}\, 1\{X_{(n)} \leq 1\}$$

$\Rightarrow$ ~~$X_{(n)} \geq 1$~~ RHS $= 0 \Rightarrow X_{(n)} \geq 1 \Rightarrow \phi = 1$.

whereas $p_{\theta_1}(X) < k\, p_{\theta_0}(X) \implies$

$\implies \mathbb{1}\{X_{(1)} \geq \theta_1\}\, \mathbb{1}\{X_{(n)} \leq \theta_1+1\} < \mathbb{1}\{X_{(1)} \geq 0\}\, \mathbb{1}\{X_{(n)} \leq 1\}$

$\implies$ ~~$X_{(n)}$~~ LHS $= 0$ and RHS $= 1$

$\implies \left\{X_{(1)} < \theta_1 \text{ or } X_{(n)} > \theta_1+1\right\}$ and $\left\{X_{(1)} \geq 0 \text{ and } X_{(n)} \leq 1\right\}$

$\implies X_{(1)} < \theta_1$ and $X_{(1)} \geq 0$ and $X_{(n)} \leq 1$

$\implies X_{(1)} < \theta_1$ ~~$\implies$~~

$\implies X_{(1)} < c$

$\implies \phi = 0$

$\therefore$ ϕ is MP by NP lemma.

Case 2: $\theta_1 \geq c$

Now choose $k = 0$, so that

$p_{\theta_1}(X) > k\, p_{\theta_0}(X) \implies p_{\theta_1}(X) > 0$

$\implies X_{(1)} \geq \theta_1 \implies X_{(1)} \geq c \implies \phi = 1$

whereas $p_{\theta_1}(X) < k\, p_{\theta_0}(X)$ never happens $\therefore$ ϕ is MP

1995 Q4

as ϕ is MP level α for $\theta=0$ against $\theta=\theta_1$ $\forall \theta_1>0$,

it follows that ϕ is UMP for $\theta=0$ vs $\theta>0$. $\Box$

(iv) From (i), ~~$c = (1-\alpha$~~ $c = 1-\alpha^{1/n}$

from (ii) $\beta(\theta) = \begin{cases} 1-(1-\theta)^n+(1-c) & \text{for } \theta\in[0,c) \\ 1 & \text{for } \theta\geq c. \end{cases}$

As stated the question is trivial, any n and c will do!

Suppose we are asking power $= 0.8$ for $\theta>\theta_1$ fixed.

then need $0.8 = 1-(1-\theta_1)^n+(1-c)$

$= 1-(1-\theta_1)^n+\alpha^{1/n}$

$\therefore \ (1-\theta_1)^n - 0.1^{1/n} = 0.2$

can solve numerically for n ...

1995 Q7

(1) Suppose not, for a contradiction.

Then, for some i, $\hat{f}_n(x)$ is a non-constant non-increasing function on $x \in (X_{(i)}, X_{(i+1)})$.

$$\therefore \int_{X_{(i)}}^{X_{(i+1)}} \hat{f}_n(x)\,dx > \hat{f}_n(X_{(i+1)})(X_{(i+1)} - X_{(i)})$$

Then
$$\tilde{f}_n(x) = \begin{cases} \hat{f}_n(x) & \forall x \notin (X_{(i)}, X_{(i+1)}] \\ \frac{1}{X_{(i+1)} - X_{(i)}} \int_{X_{(i)}}^{X_{(i+1)}} \hat{f}_n(x)\,dx & \forall x \in (X_{(i)}, X_{(i+1)}] \end{cases}$$

Achieves a greater product $\prod \tilde{f}_n(X_i) > \prod \hat{f}_n(X_i)$ $\times$.

(2) In this setting, $\prod_{i=1}^{n} g_n(X_i) = \prod_{j=1}^{k} (c_j)^{\#\{i : X_i \in A_j\}} = \prod_{j=1}^{k} c_j^{n\mu_n(A_j)}$

Consider some alternative distn. $\tilde{g}_n$ s.t. $\tilde{g}_n = \tilde{c}_j$ on A_j.

Then $\int_0^1 \tilde{g}_n(x)\,dx = 1 \Rightarrow \sum_{j=1}^{k} \tilde{c}_j \lambda(A_j) = 1$.

But $\prod_{i=1}^{n} \hat{g}_n(X_i) \geq \prod_{i=1}^{n} \tilde{g}_n(X_i)$

iff $\prod_{j=1}^{k} \left(\frac{\mu_n(A_j)}{\lambda(A_j)}\right)^{n\mu_n(A_j)} \geq \prod_{j=1}^{k} \tilde{c}_j^{\,n\mu_n(A_j)}$

iff $\sum_{j=1}^{k} n\mu_n(A_j) \log \frac{\mu_n(A_j)}{\lambda(A_j)} \geq \sum_{j=1}^{k} n\mu_n(A_j) \log \tilde{c}_j$

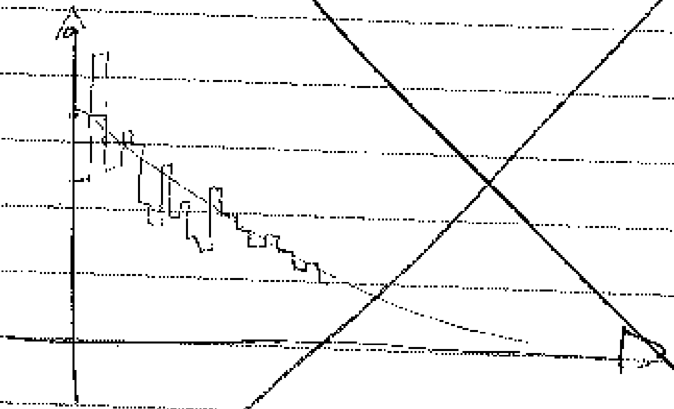
iff $\sum_{j=1}^{k} \mu_n(A_j) \log \frac{\mu_n(A_j)}{(\tilde{c}_j \lambda(A_j))} \geq 0$

which holds true by the hint. ☐

a y c_1

If   If   a

If $\frac{1}{\lambda(A_1)} < \frac{1}{\lambda(A_2)}$ then $\tilde{c}_1 = \tilde{c}_2$



$Q_n(A_n) \to 0$

$N: \forall n > N \quad \| \rho^{X_n} - \rho^{Y_n} \| < \varepsilon$.

1993 Q7

(3) Let $\hat{F}_n$ denote the smallest concave majorant of F_n (the empirical cdf), and let $\hat{f}_n$ denote the associated pdf.

By part (1), it suffices to show that

$$\sum_{i=1}^{n} \log \tilde{f}_n(X_{(i)}) \le \sum_{i=1}^{n} \log \hat{f}_n(X_{(i)}),$$

where $\tilde{f}_n$ is any other ~~step density~~ decreasing step density with jumps at the order statistics. Now note

$$\frac{\text{LHS}}{n} = \sum_{i=1}^{n} \log \tilde{f}_n(X_{(i)}) \overbrace{(F_n(X_{(i)}) - F_n(X_{(i-1)}))}^{1/n} \qquad (X_{(0)} := 0)$$

$$= \sum_{i=1}^{n-1} \overbrace{\left[\log \tilde{f}_n(X_{(i)}) - \log \tilde{f}_n(X_{(i+1)})\right]}^{\ge 0 \text{ as } \tilde{f}_n \text{ is decreasing}} F_n(X_{(i)}) + \log \tilde{f}_n(X_{(n)})$$

$$\le \sum_{i=1}^{n-1} \left[\qquad \text{ditto} \qquad \right] \hat{F}_n(X_{(i)}) + \log \tilde{f}_n(X_{(n)}) \qquad (\hat{F}_n(X_{(i)}) \ge F_n(X_{(i)}))$$

$$= \sum_{i=1}^{n} \log \tilde{f}_n(X_{(i)}) (\hat{F}_n(X_{(i)}) - \hat{F}_n(X_{(i-1)}))$$

$$\le \sum_{i=1}^{n} \log \hat{f}_n(X_{(i)}) (\hat{F}_n(X_{(i)}) - \hat{F}_n(X_{(i-1)}))$$

where the last inequality follows because

$$\sum_{i=1}^{n} \log\left(\frac{\hat{f}_n(X_{(i)})}{\tilde{f}_n(X_{(i)})}\right) \underbrace{(\hat{F}_n(X_{(i)}) - \hat{F}_n(X_{(i-1)}))}$$

$$= \sum_{i=1}^{n} \left(\hat{f}_n(X_{(i)})(X_{(i)} - X_{(i-1)})\right) \log\left(\frac{\hat{f}_n(X_{(i)})(X_{(i)} - X_{(i-1)})}{\tilde{f}_n(X_{(i)})(X_{(i)} - X_{(i-1)})}\right) \ge 0$$

by the hint from part (2), as

$$\sum_{i=1}^{n} f_n(X_{(i)})(X_{(i)} - X_{(i-1)}) = \sum_{i=1}^{n} \left(F_n(X_{(i)}) - F_n(X_{(i-1)}) \right) = 1.$$

Lastly, note that

$$\sum_{i=1}^{n} \log \hat{f}_n(X_{(i)}) \left(\hat{F}_n(X_{(i)}) - \hat{F}_n(X_{(i-1)}) \right) =$$

$$= \sum_{i=1}^{n} \log \hat{f}_n(X_{(i)}) \left(F_n(X_{(i)}) - F_n(X_{(i-1)}) \right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \log \hat{f}_n(X_{(i)})$$

since ~~all~~ at the points of non-constancy of $\hat{f}_n(X_{(i)})$,

we have that $\hat{F}_n(X_{(i)}) = F_n(X_{(i)})$.

Hence $\frac{1}{n}$ LHS $\leq \frac{1}{n}$ RHS

$\therefore$ LHS $\leq$ RHS as required. $\square$

1994 Q2

(1) $f(y|\theta) = c(\theta) g(y) \mathbb{1}\{y \in [0, b(\theta)]\}$

So the ≡ likelihood is

$$L(\theta; y) = \prod_{i=1}^{n} c(\theta) g(y_i) \mathbb{1}\{y_i \in [0, b(\theta)]\}$$

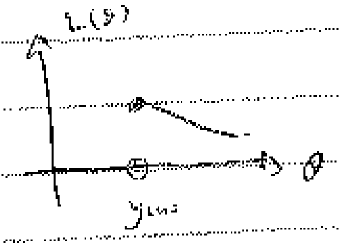
$$= c(\theta)^n \cancel{g(y_i)} \prod_i g(y_i) \prod_i \mathbb{1}\{y_i \le b(\theta)\}$$

$$= c(\theta)^n \mathbb{1}\{b(\theta) \ge y_{(n)}\} \prod_i g(y_i)$$

$$= c(\theta)^n \mathbb{1}\{\theta \ge b^{-1}(y_{(n)})\} \prod_i g(y_i)$$

as ~~$b(\theta) \ge 0$~~ $b(\theta)$ is increasing so b^{-1} exists.

therefore, if $c(\theta)$ is decreasing, $\hat{\theta}_{MLE} = b^{-1}(y_{(n)})$



But, if $\theta_1 < \theta_2$, then $b(\theta_1) < b(\theta_2)$, so that

$$\int_0^{b(\theta_1)} g(y)\,dy \le \int_0^{b(\theta_2)} g(y)\,dy \quad \therefore \frac{1}{c(\theta_1)} \le \frac{1}{c(\theta_2)}$$

$\therefore$ ~~$c(\theta_1)$~~ $c(\theta_1) \ge c(\theta_2)$ with strict inequality if $g > 0$.

Thus $\hat{\theta}_{MLE} = b^{-1}(Y_{(n)})$ is a maximizer of $L(\theta;X)$,

and is in fact the UNIQUE maximizer when $g > 0$

(otherwise MLE does not exist as there ~~cant~~ are multiple maximizers).

By Neyman-Fisher factorization criterion, $Y_{(n)}$ is sufficient for θ.

(2) $$\Lambda(Y) = \frac{\cancel{\sup} L(\theta_0;Y)}{\sup_\theta L(\theta;Y)}$$

$$= \frac{L(\theta_0,Y)}{L(\hat{\theta}_{MLE},Y)}$$

$$= \frac{c(\theta_0)^n \mathbb{1}\{b(\theta_0) \geq Y_{(n)}\}}{c(b^{-1}(Y_{(n)}))^n \mathbb{1}\{b(\cancel{X}b^{-1}(Y_{(n)})) \geq Y_{(n)}\}}$$

$$= \left(\frac{c(\theta_0)}{c(b^{-1}(Y_{(n)}))}\right)^n \mathbb{1}\{b(\theta_0) \geq Y_{(n)}\}$$

If $Y_{(n)} > b(\theta_0)$, this is $= 0$, otherwise

$$\therefore -2\log\Lambda(Y) = -2n\log\left(c(\theta_0)/c(b^{-1}(Y_{(n)}))\right)$$

But recall that $c(\theta)$ is the normalising constant $c(\theta) = \frac{1}{\int_0^{b(\theta)} g(y)\,dy}$

$$\therefore c(b^{-1}(Y_{(n)})) = \frac{1}{\int_0^{Y_{(n)}} g(y)\,dy}$$

$$\therefore W = -2\log\Lambda(Y) = -2n\log\int_0^{Y_{(n)}} c(\theta_0)\,g(y)\,dy$$ as required.

1994 Q2

(3) Suppose $Y_1, \ldots, Y_n \sim f(y|\theta_0)$

Then $P(W \leq z) = P\left(-2n\log\int_0^{Y_{(n)}} c(\theta_0)g(y)\,dy \leq z\right)$

$$= P\left(n\log\int_0^{Y_{(n)}} c(\theta_0)g(y)\,dy \geq -\frac{z}{2}\right)$$

$$= P\left(\left(\int_0^{Y_{(n)}} c(\theta_0)g(y)\,dy\right)^n \geq e^{-\frac{z}{2}}\right)$$

$$= P\left(\int_0^{Y_{(n)}} c(\theta_0)g(y)\,dy \geq e^{-\frac{z}{2}\cdot\frac{1}{n}}\right)$$

$$= P\left(\int_0^{Y_{(n)}} c(\theta_0)g(y)\,dy \geq \int_0^{q} c(\theta_0)g(y)\,dy\right)$$ where (q is the $e^{-\frac{z}{2}\frac{1}{n}}$ quantile of Y_0)

$$= P\left(Y_{(n)} \geq q\right)$$

$$= 1 - P\left(Y_{(n)} < q\right)$$

$$= 1 - \left(\int_0^{q} c(\theta_0)g(y)\,dy\right)^n \quad \text{(independence)}$$

$$= 1 - \left(e^{-\frac{z}{2}\cdot\frac{1}{n}}\right)^n$$

$$= 1 - e^{-\frac{z}{2}}$$ as required.

1994 Q3

(1) CLT: $\sqrt{n}\,(F_n(x) - F_0(x)) \xrightarrow{d} N(0, F_0(x)(1-F_0(x)))$ $\square$

(2) Let $U_i = F_0(X_i)$ $\forall i$, so that $U_1, \dots, U_n \sim \mathcal{U}(0,1)$.

$\therefore$ $F_n(x) = \frac{1}{n}\sum_1^n \mathbb{1}_{\{X_i \le x\}} = \frac{1}{n}\sum_1^n \mathbb{1}_{\{F(X_i) \le F_0(x)\}} = \tilde{F}_n(F_0(x))$

where $\tilde{F}_n(u) = \frac{1}{n}\sum_1^n \mathbb{1}_{\{U_i \le u\}}$.

$\therefore$ $\sup_{x\in\mathbb{R}} |F_n(x) - F_0(x)| = \sup_{u\in[0,1]} |\tilde{F}_n(u) - u|$ which depends only

on a sequence of n $\mathcal{U}(0,1)$ RVs, i.e. it is free of F_0. $\square$

(3) Let $\varepsilon > 0$. Fix an integer $K > \frac{1}{\varepsilon}$.

$\forall u \in [0,1]$, $u \in \left[\frac{k-1}{K}, \frac{k}{K}\right]$ for some $k \in \{1, 2, \dots, K\}$

$\therefore$ $\tilde{F}_n(u) - u \le \tilde{F}_n(\frac{k}{K}) - \frac{k-1}{K} = \left(\tilde{F}_n(\frac{k}{K}) - \frac{k}{K}\right) + \frac{1}{K}$

$\tilde{F}_n(u) - u \ge \tilde{F}_n(\frac{k-1}{K}) - \frac{k}{K} = \left(\tilde{F}_n(\frac{k-1}{K}) - \frac{k-1}{K}\right) + \frac{1}{K}$

$\left(\begin{array}{l} F_n(x) - F_0(x) \le F_n(t_k) - F_0(t_{k-1}) = \left(F_n(t_k) - F_0(t_k)\right) + \frac{1}{K} \\ F_n(x) - F_0(x) \ge F_n(t_{k-1}) - F_0(t_k) = F_n(t_{k-1}) - F_0(t_{k-1}) + \frac{1}{K} \end{array} \right)$

$\therefore$ $\sup_{u\in[0,1]} |\tilde{F}_n(u) - u| \le \max_{k\in\{0,\dots,K\}} \left|\tilde{F}_n(\frac{k}{K}) - \frac{k}{K}\right| + \frac{1}{K}$

But, by SLLN, $\max_{k\in\{0,\dots,K\}} \left|\tilde{F}_n(\frac{k}{K}) - \frac{k}{K}\right| \longrightarrow 0$ a.s.

$\therefore$ $\limsup_{n\to\infty} \sup_{u\in[0,1]} |\tilde{F}_n(u) - u| \le \frac{1}{K}$ a.s. $\forall K \in \mathbb{Z}^+$.

$\therefore$ $P\left(\limsup \sup_{u\in[0,1]} |\tilde{F}_n(u) - u| < \varepsilon \;\; \forall \varepsilon > 0\right) = 1$ $\square$

$\left(P\left(\limsup |\tilde{F}_n(u) - u| > \frac{1}{K} \text{ for some } K\right) \le \sum_{K=1}^{\infty} P\left(\limsup |\tilde{F}_n(u) - u| > \frac{1}{K}\right) = 0 \right)$

1999 Q4

(1) We are given $X_1,\dots,X_n \overset{iid}{\sim} N(r,\sigma^2)$ $r=\mu$

Area of circle $=$ ~~8~~ πr^2.

$E X_1^2 = \text{Var}\, X_1 + E^2 X_1 = \sigma^2 + r^2$

~~$E\bar{X}$~~ and $S^2 = \frac{1}{n-1}\sum_1^n (X_i-\bar{X})^2$ is an

unbiased estimator of σ^2

Thus $\pi(X_1^2 - S^2)$ is an unbiased estimator of the area.

(2) $L(\mu,\sigma^2;X) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum(X_i-\mu)^2\right\}$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum X_i^2 + \frac{\mu}{\sigma^2}\sum X_i - \frac{n\mu^2}{2\sigma^2}\right\}$$

This is a 2-parameter exponential family with

~~st~~ $\eta_1 = -\frac{1}{2\sigma^2}$, $\eta_2 = \frac{\mu}{\sigma^2}$, $T_1 = \sum X_i^2$, $T_2 = \sum X_i$

and parameter space $\eta = \mathbb{R}^- \times \mathbb{R} = (-\infty,0)\times\mathbb{R}$

which has non-empty interior. By class results,

$(\sum X_i, \sum X_i^2)$ is MS and CS for the family.

∴ $\pi(\bar{X}^2 - \frac{1}{n}S^2)$ is an unbiased funct of CS stats, an

$$E\bar{X}^2 = \frac{1}{n^2}E(\Sigma X_i)^2 = \cancel{\frac{1}{n}EX_1^2}\ \frac{1}{n^2}E\left(N(n\mu, n\sigma^2)\right)^2$$

$$= \frac{1}{n^2}\left[n\sigma^2 + n^2\mu^2\right] = \frac{1}{n}\sigma^2 + \mu^2 \qquad ①$$

and $s^2 = \frac{1}{n-1}\left(\Sigma X_i^2 - n\bar{X}^2\right)$

$\therefore$ UMVUE is $\pi\left(\bar{X}^2 - \frac{1}{n}s^2\right)$

(3) MLE for r is $\bar{X}$

$\therefore$ MLE for r^2 is $\bar{X}^2$ (invariance of MLE)

By I this is biased.

(4) let $R_n = \pi\bar{X}^2$ (MLE)

$T_n = \pi\left(\bar{X}^2 - \frac{1}{n}s^2\right)$ (UMVUE)

By CLT, $\sqrt{n}(\bar{X} - r) \xrightarrow{d} N(0, \sigma^2)$

By Δ-method, $\sqrt{n}\left(R_n - \pi r^2\right) \xrightarrow{d} N(0, 4\pi^2 r^2\sigma^2)$ Ⅱ

(with $g(\mu) = \pi\mu^2 \therefore g'(\mu) = 2\pi\mu$)

On the other hand, UMVUE:

$\sqrt{n}(T_n - \pi r^2) = \sqrt{n}(R_n - \pi r^2) + \frac{1}{\sqrt{n}}s^2 \xrightarrow{d} N(0, \cancel{8}4\pi^2 r^2\sigma^2)$

~~$\sqrt{n}T_n = \sqrt{n}\pi(\bar{X}^2) - \frac{\pi}{\sqrt{n}}s^2 \xrightarrow{d} N(0, 9\pi^2 r^2\sigma^2)$ as~~

by Slutsky's, using Ⅱ and $s^2 = O_p(1) \therefore \frac{1}{\sqrt{n}}s^2 = o_p(1)$

~~$= \pi\left(\frac{1}{n^2}(\Sigma X_i)^2 - \frac{1}{n}\frac{1}{(n-1)}\left(\Sigma X_i^2 - \frac{1}{n}(\Sigma X_i)^2\right)\right)$~~

$\therefore$ Both estimators are equally good asymptotically. Relative asymptotic

1994 Q8

(1) $P(Y=y) = P(X=y \mid X \geq 1) = \dfrac{P(X=y, X\geq 1)}{P(X\geq 1)} = \dfrac{\binom{n}{y} p^y (1-p)^{n-y}}{1-(1-p)^n}$ for $y = 1, 2, \ldots, n$.

$\therefore L(p;y) = \exp\left\{ y \log \frac{p}{1-p} + n\log(1-p) - \log(1-(1-p)^n) \right\} \binom{n}{y}$

this is an exponential family with natural parameter

$\eta = \log \frac{p}{1-p}$, $T(y) = y$, $\bar{\eta} = \mathbb{R}$

By class results, $T(Y) = Y$ is C.S.

(2) Compute

$$EY = \sum_{y=1}^{n} y \frac{\binom{n}{y} p^y (1-p)^{n-y}}{1-(1-p)^n} = \frac{1}{1-(1-p)^n} \sum_{y=0}^{n} y \binom{n}{y} p^y (1-p)^{n-y}$$

$$= \frac{1}{1-(1-p)^n} E\,\mathrm{Bin}(n,p) = \frac{p}{1-(1-p)}$$

Hence Y is the UMVUE (unbiased func of C.S. statistic.)

(3) Yes it is unique (standard fact - by class results)

Suppose $\exists\, T$ some other UMVUE. Then T is a function of the C.S. statistic Y by Rao-Blackwell §

$$\therefore E\,T-Y = ET - EY = \frac{p}{1-(1-p)^n} - \frac{p}{1-(1-p)^n} = 0 \quad \forall p$$

$\therefore T = Y$ a.s. $\square$

1992 Q1

[illegible] 1992 Q1

$$L(a,b,\sigma^2; X,Y) = (2\pi\sigma^2)^{-\frac{2n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\Sigma(X_i-a)^2 - \frac{1}{2\sigma^2}\Sigma(Y_i-b)^2\right)$$

$$= (2\pi\sigma^2)^{-n} \exp\left\{-\frac{1}{2\sigma^2}\left(\Sigma(X_i^2+Y_i^2)\right) + \frac{a}{\sigma^2}\Sigma X_i + \frac{b}{\sigma^2}\Sigma Y_i - \frac{na^2+nb^2}{2\sigma^2}\right)$$

Recognise exp. fam. $\theta = (a,b,\sigma^2)$

$$\eta(\theta) = \left(-\frac{1}{2\sigma^2}, \frac{a}{\sigma^2}, \frac{b}{\sigma^2}\right), \quad \bar{\eta} = (-\infty,0)\times\mathbb{R}\times\mathbb{R}$$

non-empty interior

$\therefore \quad T = \left(\Sigma X_i^2+Y_i^2, \Sigma X_i, \Sigma Y_i\right)$ is C.S.

By class results $\frac{1}{n-1}\Sigma(X_i-\bar{X})$ is unbiased for σ^2

$\therefore \quad \frac{1}{n-1}\left(\Sigma X_i^2 - n\bar{X}^2\right)$ "

$\therefore \quad E\ \Sigma X_i^2 - n\bar{X}^2 = (n-1)\sigma^2$

$E\ \Sigma Y_i^2 - n\bar{Y}^2 = (n-1)\sigma^2$

$\therefore \quad E\ \Sigma(X_i^2+Y_i^2) - n\bar{X}^2 - n\bar{Y}^2 = (2n-2)\sigma^2$

$$\therefore \quad \hat{\sigma}^2 = \frac{1}{2n-2}\left[\sum_1^n (X_i^2+Y_i^2) - n\bar{X}^2 - n\bar{Y}^2\right]$$

$$= \frac{1}{2n-2}\left[\sum_1^n (X_i-\bar{X})^2 + (Y_i-\bar{Y})^2\right)$$ is UMVUE for σ^2.

As for $\frac{a-b}{\sigma^2}$, idea: $\bar X - \bar Y$ is UMVUE for $a-b$.

$\bar X - \bar Y$ is unbiased for $a-b$.

By class results; $\Sigma (X_i-\bar X)^2 \sim \sigma^2 \chi^2_{n-1}$

$\Sigma (Y_i-\bar Y)^2 \sim \sigma^2 \chi^2_{n-1}$

and the two samples are independent

$\therefore \quad \Sigma (X_i-\bar X)^2 + \Sigma (Y_i-\bar Y)^2 \sim \sigma^2 \chi^2_{2n-2}$ $\quad\therefore$ this is ancillary for (a,b)

$$\therefore \quad \frac{1}{\sigma^2}\left[\Sigma (X_i-\bar X)^2 + \Sigma (Y_i-\bar Y)^2 \right] = U \sim \chi^2_{2n-2}$$

$$\therefore \quad E\, U^{-\frac{1}{2}} = \frac{\Gamma((2n-2-1)/2)}{\Gamma((2n-2)/2)}\, 2^{-\frac{1}{2}}$$

$$\therefore \quad E\, 1 \Big/ \sqrt{\Sigma (X_i-\bar X)^2 + \Sigma (Y_i-\bar Y)^2} = \sigma^{-1} \left(\frac{\Gamma\left(\frac{2n-3}{2}\right)}{\Gamma(n-1)}\, 2^{-\frac{1}{2}} \right)$$

$\therefore \quad \dfrac{2^{\frac{1}{2}}\, \Gamma(n-1)}{\Gamma\left(\frac{2n-3}{2}\right)} \Big/ \sqrt{\Sigma (X_i-\bar X)^2 + \Sigma (Y_i-\bar Y)^2}$ is unbiased for $\dfrac{1}{\sigma}$

$\therefore \quad \dfrac{2^{\frac{1}{2}}\, \Gamma(n-1)}{\Gamma\left(\frac{2n-3}{2}\right) \sqrt{\Sigma (X_i-\bar X)^2 + \Sigma (Y_i-\bar Y)^2}}$ is unbiased for $\dfrac{1}{\sigma}$.

Lastly, by Basu's, $\overbrace{(\bar X, \bar Y)}^{\text{C.S. for } (a,b)} \perp\!\!\!\perp \left(\overbrace{\sqrt{\Sigma (X_i-\bar X)^2 + \Sigma (Y_i-\bar Y)^2}}^{\text{ancillary for } (a,b)} \right)$ (ancillary as $\sigma = \sigma_0$ known)

$\therefore \quad \dfrac{\sqrt{2}\, \Gamma(n-1)\, (\bar X - \bar Y)}{\Gamma\left(\frac{2n-3}{2}\right) \sqrt{\Sigma (X_i-\bar X)^2 + \Sigma (Y_i-\bar Y)^2}}$ is UMVUE (unbiased func of C.S. statistic)

1992 Q2

The likelihood is $p_\theta(x) = \prod_1^n \frac{1}{\theta} \mathbb{1}\{x_i \le \theta\} = \theta^{-n} \mathbb{1}\{x_{(n)} \le \theta\}$

(i) Fix $\theta_1 > \theta_0$. By NP lemma, $\exists$ MP test ϕ for $\theta = \theta_0$ vs $\theta = \theta_1$, s.t.

$$\phi(X) = \begin{matrix} 1 & \text{if } p_{\theta_1}(X) > k\, p_{\theta_0}(X) \\ 0 & \text{if } p_{\theta_1}(X) < k\, p_{\theta_0}(X) \end{matrix}$$

$$E_{\theta_0}\phi(X) = \frac{1}{\theta_0^n}.$$

$\therefore$ ~~letting $\alpha = \frac{1}{\theta_0}$,~~ we have ~~$P_{\theta_0}$~~

$$P_{\theta_0}(p_{\theta_1}(X) > k\, p_{\theta_0}(X)) = P_{\theta_0}(\theta_1^{-n} > k\, \theta_0^{-n}) = P_{\theta_0}\left(\left(\frac{\theta_0}{\theta_1}\right)^n > k\right)$$

which is equal to 1 if $k < \left(\frac{\theta_0}{\theta_1}\right)^n$

$\therefore$ $k \ge \left(\frac{\theta_0}{\theta_1}\right)^n$, as $E_{\theta_0}\phi(X) \ge P_{\theta_0}(p_{\theta_1}(X) > k\, p_{\theta_0}(X))$

On the other hand, if $k > \left(\frac{\theta_0}{\theta_1}\right)^n$, then

$P_{\theta_0}(p_{\theta_1}(X) < k\, p_{\theta_0}(X)) = 1$, $\therefore$ $E_{\theta_0}\phi(X) = 0 \ne \left(\frac{1}{\theta_0}\right)^n$.

Hence, the only possible value of k is $k = \left(\frac{\theta_0}{\theta_1}\right)^n$.

in which case our test becomes:

$$\phi(X) = \begin{matrix} 1 & \text{if } \theta_0 < X_{(n)} \le \theta_1 \\ 0 & \text{if } \mathbb{1}\{x_{(n)} \le \theta_1\} < \mathbb{1}\{x_{(n)} \le \theta_0\} \end{matrix} \quad \text{(never happens)}$$

Thus, one possible level $1/\theta_0^n$ test is:

$$\phi(X) = \begin{cases} 1 & \text{if } X_{(n)} > \theta_0 \\ 1/\theta_0^n & \text{if } X_{(n)} \le \theta_0 \end{cases}$$

By NP lemma this is ~~UMP~~ MP for $\theta = \theta_0$ vs $\theta = \theta_1$.

But it is also free of the alternative θ_1.

$\therefore$ $\phi(X)$ is UMP for ~~this p~~ $\theta = \theta_0$ vs ~~$\theta \geq$~~ $\theta > \theta_0$.

Alternatively, argue we have MLR in $X_{(n)}$ and apply class results.

(ii) Fix $\theta_1 < \theta_0$. Consider the test

$$\phi(X) = \begin{matrix} 1 & \text{if } X_{(n)} = 1 \\ 0 & \text{o/w} \end{matrix}$$

Then $E_{\theta_0} \phi(X) = P_{\theta_0}(X_{(n)} = 1) = (1/\theta_0)^n$

and also $\phi(X)$ satisfies the NP lemma for $k = \left(\frac{\theta_0}{\theta_1}\right)^n$;

~~$\phi(X) =$~~ $f_{\theta_1}(X) > k f_{\theta_0}(X) \Rightarrow \mathbb{1}_{\{X_{(n)} \le \theta_1\}} \le \mathbb{1}_{\{X_{(n)} \le \theta_0\}}$

which never happens; $f_{\theta_1}(X) < k f_{\theta_0}(X) \Rightarrow X_{(n)} > \theta_1$

$\Rightarrow X_{(n)} > 1 \Rightarrow \phi(X) = 1$.

$\therefore$ By NP lemma, ϕ is MP for $\theta = \theta_0$ vs $\theta = \theta_1$.

But ϕ is free of the alternative.

$\therefore$ ϕ is UMP □

(iii) The test in (ii) also works for (i) $\therefore$ it is UMP □

1992 Q4

$$L(\mu,\sigma^2;X,Y) = (2\pi\sigma^2)^{-\frac{2n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_i)^2+(y_i-\mu_i)^2\right\}$$

$$= (2\pi\sigma^2)^{-n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(x_i-\frac{x_i+y_i}{2}\right)^2+\left(y_i-\frac{x_i+y_i}{2}\right)^2+2\left(\frac{x_i+y_i}{2}-\mu_i\right)^2\right\}$$

$$= (2\pi\sigma^2)^{-n} \exp\left\{-\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{x_i-y_i}{2}\right)^2+\left(\frac{x_i+y_i}{2}-\mu_i\right)^2\right\}$$

Ⓘ $\therefore\ l(\mu,\sigma^2;X,Y) = k - n\log\sigma^2 - \frac{1}{\sigma^2}\sum\left(\frac{x_i-y_i}{2}\right)^2 - \frac{1}{\sigma^2}\sum\left(\frac{x_i+y_i}{2}-\mu_i\right)^2$

Now note $\sum\left(\frac{x_i+y_i}{2}-\mu_i\right)^2 \geq 0$,

and $\sum\left(\frac{x_i-y_i}{2}\right)^2 > 0$ a.s

$\therefore\ l(\mu,\sigma^2;X,Y) \longrightarrow -\infty$ as $\sigma^2 \longrightarrow \infty$ or $\sigma^2 \longrightarrow 0$.

Therefore, the unique stationary point $\hat{\sigma}^2 = \frac{\sum(x_i-\hat{\mu}_i)^2+(y_i-\hat{\mu}_i)^2}{2n}$

is the global maximiser in σ^2.

Also from I, the MLE in $\vec{\mu}$ is $\hat{\mu}_i = \frac{x_i+y_i}{2}$ (optimality - product

$\therefore\ \hat{\sigma}^2_{MLE} = \frac{\sum(x_i-y_i)^2}{4n}$

Note that $z_i := X_i - Y_i = N(0,2\sigma^2)$ $\therefore$ $z_i^2 = (X_i-Y_i)^2 \sim 2\sigma^2\chi^2_1$

∴ By WLLN, $\hat{\sigma}^2_{MLE} \xrightarrow{P} \frac{1}{4} E Z_1^2 = \frac{2\sigma^2}{4} = \frac{\sigma^2}{2}$

∴ $\hat{\sigma}^2_{MLE}$ is NOT consistent.

Consistent estimator would be

~~$\hat{\sigma}^2_{MLE}$~~ $\hat{\sigma}^2 = 2\hat{\sigma}^2_{MLE}$ instead.

1992 Q5

$$p(x;\theta) = \exp\left\{ c(\theta)\sum_1^n T(x_i) + nd(\theta) + \sum_1^n S(x_i)\right\}$$

$$\ell(\theta;x) = c(\theta)\sum_1^n T(x_i) + nd(\theta) + \sum_1^n S(x_i)$$

$$\therefore \quad \ell'(\theta;x) = c'(\theta)\sum T(x_i) + nd'(\theta)$$

$$\ell''(\theta;x) = c''(\theta)\sum T(x_i) + nd''(\theta)$$

Therefore $\ell'(\theta;x) = 0 \implies \frac{1}{n}\sum T(x_i) = -\frac{d'(\theta)}{c'(\theta)}$

Let $\eta = c(\theta)$. Then $d(\theta) = A(\eta)$ for some function A.
to see this, note that

$$c(\theta_1) = c(\theta_2) \implies \int e^{c(\theta_1)T(x)+d(\theta_1)+S(x)}\,dx = \int e^{c(\theta_2)T(x)+d(\theta_2)+S(x)}\,dx$$
$$= \int e^{c(\theta)T(x)+d(\theta_2)+S(x)}\,dx$$

$$\implies e^{d(\theta_1)} = e^{d(\theta_2)}$$
$$\implies d(\theta_1) = d(\theta_2)$$

Thus $\ell(\eta;x) = \eta\sum T(x_i) + nA(\eta) + \sum S(x_i)$

$$\therefore \frac{\partial \ell}{\partial \eta} = \sum T(x_i) + nA'(\eta)$$

$$\therefore \frac{\partial^2 \ell}{\partial \eta^2} = nA''(\eta) < 0 \quad \text{as } -A''(\eta) = \operatorname{Var} T(X)$$ (class result using regularity condition)

Therefore, provided the parameter space is nice enough, the likelihood is maximized ~~at~~ at η s.t.

$$\cancel{\frac{1}{n}\Sigma X} \qquad -A'(\eta) = \frac{1}{n}\Sigma T(x_i)$$

But $A(\eta) = d(\theta)$

$$\therefore \frac{dA}{d\eta} = \frac{d}{d\eta}\left(d(\theta)\right)$$

$$= \frac{d}{d\theta}\left(d(\theta)\right) \cdot \frac{d\theta}{d\eta}$$

$$= \frac{d'(\theta)}{c'(\theta)}$$

and our ~~equation~~ likelihood is therefore maximized at θ s.t.

$$-\frac{d'(\theta)}{c'(\theta)} = \frac{1}{n}\Sigma T(X_i) \qquad \textcircled{I}$$

In the parameter ~~of~~ $\xi = -\frac{d'(\theta)}{c'(\theta)}$, this is a unique maximiser, and so $\frac{1}{n}\Sigma T(X_i)$ is MLE for ξ.

However, there may be several solns in θ to I,

so ~~this is~~ any such $\hat{\theta}$ is not necessarily the MLE

e.g. ~~$p(x)$~~ $X \sim N(0, \sigma^2)$, $\sigma \in \mathbb{R}$ $\qquad \theta = \sigma$

$$p(x;\theta) = \exp\left\{-\frac{1}{2\sigma^2}\Sigma X_i^2 - \frac{n}{2}\log\sigma^2 - \frac{n}{2}\log 2\pi\right\}$$

$\therefore -$ [illegible]

1992 . Q5

$$\therefore \quad d'(\theta) = \frac{d}{d\theta}\left(-\frac{\log\theta^2}{2}\right)$$

$$= -\frac{2\theta}{2\theta^2}$$

$$= -\frac{1}{\theta}$$

$$c'(\theta) = \frac{d}{d\theta}\left(-\frac{1}{2\theta^2}\right)$$

$$= \frac{1}{\theta^3}$$

$\therefore -\frac{d'(\theta)}{c'(\theta)} = \theta^2$ with MLE $\hat{\theta}^2 = \frac{1}{n}\sum X_i^2$

however $\hat{\theta} = \pm\sqrt{\frac{1}{n}\sum X_i^2}$ both maximize ~~f~~ $l(x;\theta)$

so the maximum is non-unique $\therefore$ NO MLE exists.

1992 Q7

(i) This is not true. For example, let $P_\theta = N(0,1)$ for $\theta \in \mathbb{R}$,

then the $\frac{p_{\theta_2}(x)}{p_{\theta_1}(x)} = 1$ which is a non-decreasing function of x.

So this family is MLR. However it is NOT stochastically increasing.

<u>Defn</u> Stochastically increasing: $\forall \theta_1 < \theta_2$, $P_{\theta_2}(X>t) \geq P_{\theta_1}(X>t)$ $\forall t$

and $P_{\theta_2}(X>t) > P_{\theta_1}(X>t)$ for some t.

However, we can prove that MLR in X $\Longrightarrow$ "weakly stochastically increasing"

Suppose $\{P_\theta, \theta \in \Theta\}$ is MLR in X.

Let $t \in \mathbb{R}$. Then $\psi(x) = \mathbb{1}\{x>t\}$ is non-decreasing.

By class results, R_θ $E_\theta \psi(X)$ is non-decreasing.

$\therefore$ $P_\theta(X>t)$ is non-decreasing. □

(ii) Counterexample: Cauchy distn.

Let $p_\theta(x) = \frac{1}{\pi} \cdot \frac{1}{1+(x-\theta)^2}$

Then $X|\theta \sim \theta + \text{Cauchy}(0,1)$, a location family,

so clearly $\{P_\theta, \theta \in \mathbb{R}\}$ is stochastically increasing.

However, this family is not MLR. For example, take $\theta_0 = 0, \theta_1 = 1$

$$\frac{p_{\theta_1}(x)}{p_{\theta_0}(x)} = \frac{1+(x-1)^2}{1+x^2}$$

$$= \frac{1+x^2-2x+1}{1+x^2}$$

$$= 1 + \frac{-2x+1}{1+x^2} \longrightarrow 1 \text{ as } x \to \pm\infty$$

but $= 2$ at $x = 0$ ∴ is not monotone in x □

(iii) $f_\theta(x)$ is MLR in X $\Longleftrightarrow$

$\Longleftrightarrow$ $\forall \theta_1 < \theta_2$, $\dfrac{g(x-\theta_2)}{g(x-\theta_1)}$ is non-decreasing in x

$\Longleftrightarrow$ $\forall \theta_1 < \theta_2$, $\exp\{\log g(x-\theta_2) - \log g(x-\theta_1)\}$ is non-decreasing in x

$\Longleftrightarrow$ $\forall \theta_1 < \theta_2$, $\log g(x-\theta_2) - \log g(x-\theta_1)$ is non-decreasing in x

$\Longleftrightarrow$ $\forall \theta_1 < \theta_2$, $\forall x_1 < x_2$, ~~$\log g(x_1 - \theta$~~

$$\log g(x_1-\theta_2) - \log g(x_1-\theta_1) \le \log g(x_2-\theta_2) - \log g(x_2-\theta_1)$$

$\Longleftrightarrow$ $\dfrac{h(x_1-\theta_1) - h(x_1-\theta_2)}{\theta_2-\theta_1} \geq \dfrac{h(x_2-\theta_1) - h(x_2-\theta_2)}{\theta_2-\theta_1}$ $(h = \log g)$

$\Longleftrightarrow$ $\forall \theta_1 \le \theta_2$ $\forall x_1 < x_2$

$\Longleftrightarrow$ $\dfrac{h(\tilde{x}_1+\Delta) - h(\tilde{x}_1)}{\Delta} \geq \dfrac{h(\tilde{x}_2+\Delta) - h(\tilde{x}_2)}{\Delta}$ $\forall \tilde{x}_1 < \tilde{x}_2, \Delta >$

$\Longleftrightarrow$ h is concave □

$$\log g(x_1-\theta_2) + \log g(x_2-\theta_1) \le \log g(x_2-\theta_2) + \log g(x_1-\theta_1)$$

$\Longleftrightarrow$ $\log g$ is concave □

1992 Q7

To see why the last implication holds, we argue:

"$\Longleftarrow$" : Suppose h is concave.

Let $\theta_1 < \theta_2$, $x_1 < x_2$.

Let $a_1 = x_1 - \theta_2$, $a_2 = x_2 - \theta_1$, $\Delta = \theta_2 - \theta_1 > 0$, so that

$$a_1 + \Delta = x_1 - \theta_1 , \quad a_2 - \Delta = x_2 - \theta_2 ,$$

and note that $a_1 < (a_1 + \Delta , a_2 - \Delta) < a_2$

Using concavity, $\lambda h(a_1) + (1-\lambda) h(a_2) \leq h(\lambda a_1 + (1-\lambda) a_2) \quad \forall \lambda \in (0,1)$

$$(1-\lambda) h(a_1) + \lambda h(a_2) \leq h((1-\lambda) a_1 + \lambda a_2)$$

Choosing $\lambda = 1 - \frac{\Delta}{a_2 - a_1}$, we find $\lambda a_1 + (1-\lambda) a_2 = a_1 + \Delta$

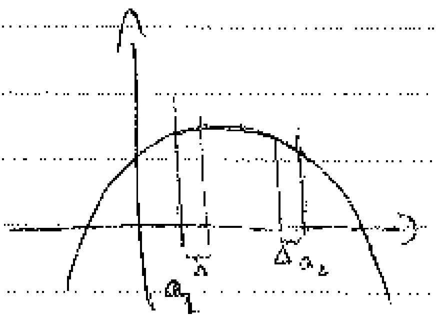
$$(1-\lambda) a_1 + \lambda a_2 = a_2 - \Delta$$

$\therefore$ adding the two inequalities, we find

$$h(a_1) + h(a_2) \leq h(a_1 + \Delta) + h(a_1 - \Delta)$$

$$\therefore \quad h(x_1 - \theta_2) + h(x_2 - \theta_1) \leq h(x_1 - \theta_1) + h(x_2 - \theta_2) \qquad \square$$

the thinking here is:



"$\Rightarrow$": Conversely, suppose $\forall \theta_1 < \theta_2$, $x_1 < x_2$,

$$h(x_1 - \theta_2) + h(x_2 - \theta_1) \leq h(x_1 - \theta_1) + h(x_2 - \theta_2)$$